1. Suppose that $A \subset X$ and $B \subset Y$. Prove that ① $cl_{X\times Y}(A\times B) = \bar{A}\times\bar{B}$ and ② $Int_{X\times Y}(A\times B) = (Int\,A)\times(In\,B)$.

① Notice that $\forall U\subset X,\ V\subset Y$ ~~open~~

$$(U\times V)\cap(A\times B) = (U\cap A)\times(V\cap B)$$

and $\bar{A} = \{x : \text{every nbd of } x \text{ intersects } A\} = \{x : \text{every open } U\ni x,\ U\cap A\neq\emptyset\}$

$\Updownarrow$

$x\in Int\,(\text{nbd})$

② $(Int\,A)\times(In\,B)$

$= (X\setminus\overline{X\setminus A})\times(Y\setminus\overline{Y\setminus B})$

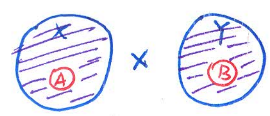
$= (X\times Y)\setminus\big((X\times\overline{Y\setminus B})\cup(\overline{X\setminus A}\times Y)\big)$ use ① and $X=\bar{X}$. $Y=\bar{Y}$

$= (X\times Y)\setminus\big(\overline{X\times(Y\setminus B)}\cup\overline{(X\setminus A)\times Y}\big)$ use $\overline{A\cup B}=\bar{A}\cup\bar{B}$

$= (X\times Y)\setminus\overline{\big((X\times(Y\setminus B))\cup((X\setminus A)\times Y)\big)}$

$= (X\times Y)\setminus\overline{(X\times Y\setminus A\times B)}$

$= Int_{X\times Y}(A\times B)$

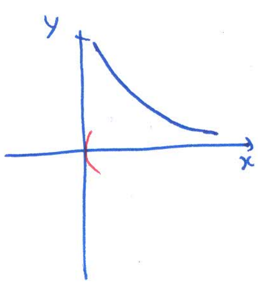


□

2. Find a continuous and open mapping which is not closed.

Projections are continuous and open.

Construct a closed subset $K\subset\mathbb{R}^2$ such that $\pi_1(K)$ is not closed in $\mathbb{R}^1$.



$K = \{(x,y)\in\mathbb{R}^2 \mid y=\frac{1}{x},\ x>0\}$.

$\pi_1(K) = (0,+\infty)$

3. Suppose that F_n is a closed subset of X_n for each $n \in \mathbb{N}$.
Prove that $\prod_{n=0}^{\infty} F_n$ is closed in the product space $\prod_{n=0}^{\infty} X_n$.
Is the same result true if we replace "closed" with "open" why?

$$x \in \prod_{n=0}^{\infty} F_n \iff \pi_n(x) \in F_n \quad \forall n \in \mathbb{N}$$

$$x \notin \prod_{n=0}^{\infty} F_n \iff \pi_k(x) \notin F_k \quad \exists k \in \mathbb{N}.$$

$$\underbrace{\left(\prod_{n=0}^{\infty} F_n\right)^c}_{\text{open}} = \bigcup_{k \in \mathbb{N}} \Big\{ \underbrace{\prod_{n=0}^{\infty} U_n}_{\mathcal{B}} \; : \; U_k = F_k^c \, , \; U_n = X_n, \; n \neq k \Big\}.$$

No. $\mathcal{B} = \{ \prod_{n=0}^{\infty} U_n \; : \; U_n$ open in X_n. $\exists k \in \mathbb{N}$. $U_n = X_n$ for $\forall n \geq k \}$.

~~Let~~ $\prod_{n=0}^{\infty} V_n$, open $V_n \subsetneq X_n$. $\forall n \in \mathbb{N}$. but $\prod_{n=0}^{\infty} V_n$ is not open in $\prod_{n=0}^{\infty} X_n$.

Every basic open sets $\prod_{n=0}^{\infty} U_n \in \mathcal{B}$. we have.

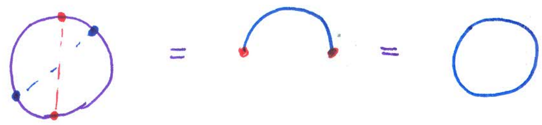
$$x \in \prod_{n=0}^{\infty} U_n \; : \; \pi_k(x) \in X_k \setminus V_n. \quad \exists k \in \mathbb{N}. \implies x \notin \prod_{n=0}^{\infty} V_n.$$

4. Projective Plane.

(a) Define $\mathbb{RP}^n$ (b) $\mathbb{RP}^1 \cong S^1$ (c) ~~RP~~ $\mathbb{RP}^2 \setminus \{$~~p~~ one point$\} \cong$ Möbius strip

(a) Define $\sim$ on $\mathbb{R}^{n+1} \setminus \{(0,\dots,0)\}$ by $v \sim w \iff \exists\, 0 \neq \lambda \in \mathbb{R}$, $v = \lambda w$

$$\mathbb{R}^{n+1} \setminus \{(0,\dots,0)\} / \sim \; = \mathbb{RP}^n$$

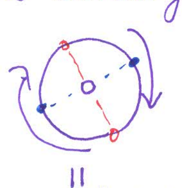
(b) $\mathbb{RP}^1 = \mathbb{R}^2 \setminus \{(0,0)\} / \sim =$  $= S^1$

(c) $\mathbb{RP}^2 \setminus \{$one point$\}$

=

~~$(D/\sim)\setminus\{$~~

delete the center of the disk with diametrically opposite points of its boundary circle identified.



=

an annulus whose outer circle is glued together at diametrically opposite points