Solutions to test 1

Q1. (a).

① Clearly, $\emptyset$ & $\mathbb{R}$ $\in \mathcal{T}_{ma}$

② If $G_\alpha \cup R_\alpha \in \mathcal{T}_{ma}$ for $G_\alpha \in \mathcal{T}_{std}$
$R_\alpha \subset A$

then $\bigcup_\alpha (G_\alpha \cup R_\alpha) = \left(\bigcup_\alpha G_\alpha\right) \cup \left(\bigcup R_\alpha\right)$

But $\mathcal{T}_{std}$ is topo $\Longrightarrow$ $\bigcup_\alpha G_\alpha \in \mathcal{T}_{std}$

$R_\alpha \subset A \Longrightarrow \bigcup_\alpha R_\alpha \subset A$

$\Longrightarrow \bigcup_\alpha (G_\alpha \cup R_\alpha) \in \mathcal{T}_{ma}$

③ If $G_1 \cup R_1$, $G_2 \cup R_2 \in \mathcal{T}_{ma}$

for $G_1, G_2 \in \mathcal{T}_{std}$, $R_1, R_2 \subset A$

Then $(G_1 \cup R_1) \cap (G_2 \cup R_2)$

$$= \underbrace{(G_1 \cap G_2)}_{\in \mathcal{T}_{std}} \cup \underbrace{(R_1 \cap G_2) \cup (R_1 \cap R_2) \cup (G_1 \cap R_2)}_{\subset A}$$

$\in \mathcal{T}_{ma}$

(b). This question is wrong.

Correct it by setting

$$\mathcal{T} = \{\phi, \mathbb{R}^2\} \cup \{ G_r : r \geq 0 \}$$

①. $\phi, \mathbb{R}^2 \in \mathcal{T}$

②. W.L.G. consider :

$G_{r_1}, \cdots, G_{r_N} \in \mathcal{T}$

$$\bigcap_{i=1}^{N} G_{r_i} = G_{\max\{r_1, \cdots, r_N\}} \in \mathcal{T}$$

③. W.L.G. consider :

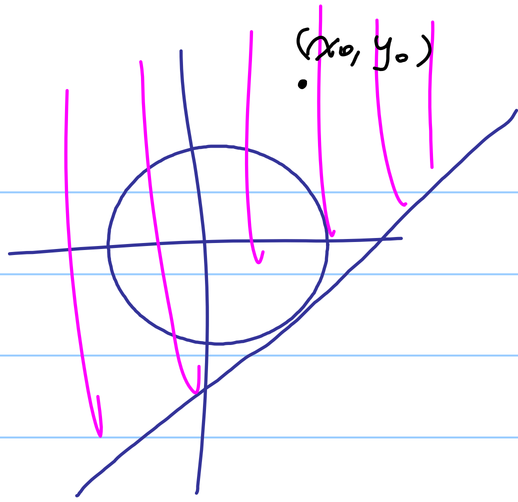
$G_{r_\alpha} \in \mathcal{T}$ for $\alpha \in I$

$$\bigcup G_{r_\alpha} = G_{\underbrace{\inf\{r_\alpha \mid \alpha \in I\}}_{\geq 0}} \in \mathcal{T}$$

(c). $\overline{C} = \{ (x, y) \mid x - y \leq \sqrt{2} \}$

Pf: ① If $(x_0, y_0) \in \{ (x, y) \mid x - y \leq \sqrt{2} \}$

If open $U \ni (x_0, y_0)$



then either $U = \mathbb{R}^2$

or $U = G_r$ for some $r \geq 0$

<u>case 1:</u> $U = \mathbb{R}^2$

$\Rightarrow U \cap C \neq \emptyset$

<u>case 2:</u> $U = G_r$

$\Rightarrow r < x_0 - y_0$

$\Longrightarrow G_r \cap C \neq \emptyset$

So $(x_0, y_0) \in \overline{C}$

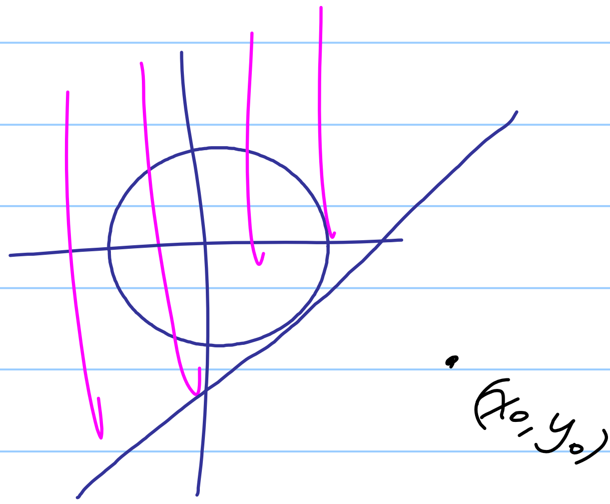
② If $(x_0, y_0) \notin \{ (x, y) \mid x - y \leq \sqrt{2} \}$

$\Rightarrow x_0 - y_0 > \sqrt{2}$

choose r_0

s.t. $\sqrt{2} < r_0 < x_0 - y_0$

then it is easy to see:



- $(x_0, y_0) \in G_{r_0}$
- $G_{r_0} \cap C = \emptyset$

$\Longrightarrow x \notin \overline{C}$

Q2. (a) See tutorial notes

(b). $\bigcap \mathcal{U} = \{x_0\}$

Pf: If $x \neq x_0$

then $x_0 \in \mathbb{R} - \{x\} \in \mathcal{T}_{cf}$

so $\exists.\ O_x \in \mathcal{U}$

s.t. $x_0 \in O_x \subset \mathbb{R} - \{x\}$

$\Longrightarrow x \notin O_x$

$\Longrightarrow x \notin \bigcap \mathcal{U}$

$\Longrightarrow \bigcap \mathcal{U} \subset \{x_0\}$

But it is obvious that $\{x_0\} \subset \bigcap \mathcal{U}$

So $\bigcap \mathcal{U} = \{x_0\}$

(c). If $x_n \not\to x$

then $\exists.$ open $U \ni x$

s.t. $\exists.\ n_1 < n_2 < \cdots$

s.t. $x_{n_k} \notin U$

for $k \geq 1$

But $\emptyset \neq U \in \mathcal{T}_{cf}$

$\Longrightarrow \quad U = \mathbb{R} - \{a_1, a_2, \cdots, a_N\}$

$\Longrightarrow \quad x_{n_k} \in \{a_1, a_2, \cdots, a_N\}$

for $\forall \; k \geq 1$

$$\Longrightarrow \quad \underbrace{\{x_{n_1}, x_{n_2}, \cdots\}}_{\text{infinite}} \subseteq \underbrace{\{a_1, a_2, \cdots, a_N\}}_{\text{finite}}$$

infinite since $\longrightarrow \; ! \longleftarrow$ finite

$x_m \neq x_n$

for $\forall \; m \neq n$

3. (a). ①. $x \in \bar{A} \iff U \cap A \neq \emptyset$ for $\forall$ open $U \ni x$

②. $x \in \operatorname{Int}(X \setminus A) \iff$

$\exists$. open $U \ni x$ s.t. $U \subseteq X \setminus A$

i.e. $U \cap A = \emptyset$

①, ② $\Longrightarrow \quad \bar{A} \cap \operatorname{Int}(X \setminus A) = \emptyset$

$\bar{A} \cup \operatorname{Int}(X \setminus A) = X$

$\Longrightarrow \quad \bar{A} = X \setminus \operatorname{Int}(X \setminus A)$

(b). ①. $A \subset Cl(A)$

$\Rightarrow \quad Cl(A) \subset Cl(Cl(A))$

②. If $x \in Cl(Cl(A))$

then for $\forall$ open $U \ni x$

we have $U \cap Cl(A) \neq \emptyset$

Suppose $y \in U \cap Cl(A)$

then $y \in U$ & $y \in Cl(A)$

i.e. U is a nbd of $y \in Cl(A)$

$\Rightarrow \quad U \cap A \neq \emptyset \quad \forall\, U$

$\Rightarrow \quad x \in Cl(A)$

(c). No.

let $X = \mathbb{R}$ w/ $\mathcal{T} = \{\emptyset, \mathbb{R}\}$

$A = \{0\}$

$A' = \mathbb{R} - \{0\}$

$\bar{A} = A \cup A' = \mathbb{R}$

$(\bar{A})' = \mathbb{R} \neq A'$

Q4. (a). (b) $\mathcal{T}_{std} \underset{①}{\subsetneq} \mathcal{T}_{lq} \underset{②}{\subsetneq} \mathcal{T}_{ll} \underset{③}{\subsetneq} \mathcal{T}_{lu}$

①. $(a, b) = \bigcup_{n=1}^{+\infty} [a+\frac{1}{n}, b)$

$[0, 1) \notin \mathcal{T}_{std}$

②. $[\sqrt{2}, 10) \in \mathcal{T}_{ll}$

$[\sqrt{2}, 10) \notin \mathcal{T}_{lq}$

Since $\sqrt{2}$ is NOT inner pt of $[\sqrt{2}, 10)$ w.r.t. $\mathcal{T}_{lq}$

③. $[a, b) = \bigcup_{n=1}^{+\infty} [a, b-\frac{1}{n}]$

$\{0\} = [-1, 0] \cap [0, 1] \in \mathcal{T}_{lu}$

$\{0\} \notin \mathcal{T}_{ll}$

(c). $f = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

5. (a). $X = \mathbb{R}^1$

$F_\alpha = \alpha$ w/ $\alpha \in I \triangleq \mathbb{R}^1$

$$f = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

(b). Consider $\forall\ x \in X$

$\forall$ open $O \ni f(x)$

Let U be open in X

s.t. $x \in U$

$U \cap F_\alpha \neq \emptyset$

only when $\alpha = \alpha_1, \cdots, \alpha_N$

For $i = 1, \cdots, N$

• case 1 $x \notin F_{\alpha_i}$

Choose open $U_i \ni x$

s.t. $U_i \cap F_{\alpha_i} = \emptyset$ (F_{α_i} closed)

• case 2 $x \in F_{\alpha_i}$

$f|_{F_\alpha}$ is cts $\Longrightarrow$ $\exists$. open $U_i \ni x$

s.t. $f(F_{\alpha_i} \cap U_i) \subset O$

Let $V \triangleq \left(\bigcap_{i=1}^{N} U_i\right) \cap U$

Claim: V is nbd of x s.t. $f(V) \subset O$

Pf: let $y \in \left(\bigcap_{i=1}^{N} U_i\right) \cap U$

$X = \bigcup F_\alpha$

$\Rightarrow \quad y \in F_{\alpha_0}$

- if $\alpha_0 \notin \{\alpha_1, \cdots, \alpha_N\}$

  $\Rightarrow \quad y \notin U \longrightarrow\longleftarrow y \in \left(\bigcap_{i=1}^{N} U_i\right) \cap U$

- if $\alpha_0 = \alpha_k$ for an $k \in \{1, \cdots, N\}$
  satisfing $x \notin F_{\alpha_k}$

  then $U_k \cap F_{\alpha_k} = \emptyset$

$$\downarrow\uparrow$$

$$y \in F_{\alpha_k} \cap U_k \neq \emptyset$$

$$\Uparrow$$

$$y \in F_{\alpha_0} = F_{\alpha_k}, \; y \in U_k$$

So $\alpha_0 = \alpha_k$ for an $k \in \{1, \dots, n\}$
satisfying $x \in F_{\alpha_k}$

$\Longrightarrow$ $y \in U_k \cap F_{\alpha_k}$

$\Longrightarrow$ $f(y) \in f(U_k \cap F_{\alpha_k}) \subseteq O$

$\uparrow$

we are in case 2