Q2.16a, Consider $f(x) = 0$ $\quad g(x) = \begin{cases} 0 & \text{if } x \le a \\ 1 & \text{if } x > a \end{cases}$

f is cont. at a,

but $f + g = g$ which is not cont. at a.

Q2.16b, Consider $f(x) = \begin{cases} 0 & \text{if } x \le a \\ 1 & \text{if } x > a \end{cases}$ $\quad g(x) = -f(x)$

both f, g discont. at a

but $f + g = 0$ which is cont at a

Q2.17 Suppose f, g are cont.

$\forall x_0$, $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t $\forall y$ which $|y - g(x_0)| < \delta_1$,

we have $|f(g(x_0)) - f(y)| < \varepsilon$

$\exists \delta_2 > 0$ s.t $\forall x$ which $|x - x_0| < \delta_2$,

we have $|g(x_0) - g(x)| < \delta_1$

$\Rightarrow$ $|f(g(x_0)) - f(g(x))| < \varepsilon$

$\Rightarrow$ $f \circ g$ is cont.

Q2.22 Ψ is cont at $x = 0$ and discont. in $\mathbb{R} \setminus \{0\}$.

claim: Ψ is cont at $x = 0$

$\forall \varepsilon > 0$, $\forall x$ which $|x - 0| < \sqrt{\varepsilon}$

$|\Psi(x) - \Psi(0)| = |x^2| < \varepsilon$

$\Rightarrow$ Ψ is cont at $x = 0$

claim: Ψ is dis-cont in $\mathbb{R} \setminus \{0\}$

$\forall x \in \mathbb{R} \setminus \{0\}$,

Case I, $x \in \mathbb{Q}$

$\because$ irrational number is dense in $\mathbb{R}$,

$\exists x_n \in (x + \frac{1}{n}, x + \frac{1}{n+1})$ where x_n is irrational

$\lim_{n \to \infty} x_n = x$

but $\Psi(x_n) = 0$

$\lim_{n \to \infty} \Psi(x_n) = 0 \neq x^2 = \Psi(x)$

Case II $x \notin \mathbb{Q}$ WLOG assume $x > 0$

$\because$ rational number is dense

$\therefore \exists\ x_n \in (x - \frac{x}{2n}, x)$ where x_n are rational

$\forall n$, $x_n > x - \frac{x}{2} = \frac{x}{2}$

$\Psi(x_n) > \frac{x^2}{4} > 0$

$\Rightarrow \Psi(x_n)$ cannot limit to 0 but $\Psi(x) = 0$

$\therefore \Psi$ is discont at $x \in \mathbb{R} \setminus \{0\}$

Q2.49a) set $N = 100$

$\forall n > N$. $|b_n - L| = \frac{1}{n+1} < \frac{1}{101} < 0.01 = \varepsilon$

Q2.49b) set $N = 1000$

$\forall n > N$ $|b_n - L| = \left|\frac{1}{n+1} - \frac{1}{1000}\right|$

$\leq \left|\frac{1}{n+1}\right| + \left|\frac{1}{1000}\right|$

$< \frac{2}{1000} < 0.01 = \varepsilon$

Q2.49c) to prove $\lim_{n\to\infty} b_n = l$, $\forall \varepsilon > 0$, we need to find N s.t $\forall n > N$, $|b_n - l| < \varepsilon$

the condition is true for a particular ε is not enough.

Q2.50) $\because a_n \to L > 0$

$L/2 > 0$, and $\exists N$ s.t $\forall n > N$

we have $|a_n - L| < L/2$

then $-L/2 < a_n - L < L/2$

$\Rightarrow L/2 < a_n < 3L/2$

Q2.52 a) true.

($\Rightarrow$) suppose $a_n \to 0$.

$\forall \varepsilon > 0\ \exists N$ s.t $\forall n > N$ $|a_n - 0| = |a_n| < \varepsilon$.

$\Rightarrow \big||a_n| - 0\big| = |a_n| < \varepsilon$

$\Rightarrow |a_n| \to 0$.

($\Leftarrow$) suppose $|a_n| \to 0$

$\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$ $\big||a_n| - 0\big| = |a_n| < \varepsilon$

$|a_n - 0| = |a_n| < \varepsilon$

$\Rightarrow$ $a_n \to 0$.

Q2.52b, wrong.

$a_n \to L \Rightarrow |a_n| \to |L|$ but $|a_n| \to |L| \not\Rightarrow a_n \to L$

see hw2 Q4.

Q2.54, $0.\bar{9} = \sum_{i=1}^{\infty} 9 \times 10^{-i}$

let $a_n = \sum_{i=1}^{n} 9 \times 10^{-i} = \frac{9}{10}\left(1 + \frac{1}{10} + \dots + \frac{1}{10^{n-1}}\right) = 1 - \left(\frac{1}{10}\right)^n$

$\forall \varepsilon > 0$, $\exists N$ s.t. $\left(\frac{1}{10}\right)^N < \varepsilon$

$\forall n > N$, $|a_n - 1| = \left(\frac{1}{10}\right)^n$

$< \left(\frac{1}{10}\right)^N < \varepsilon$

$\therefore \lim_{n\to\infty} a_n = 1 \Rightarrow 0.\bar{9} = 1$

Q2.64a, $\forall x \in [0, 1)$

$f_n(x) = x^n \to 0$

when $x = 1$, $f_n(x) = 1 \to 1$.

$\therefore$ f_n converge ptwise in $[0, 1]$

Q2.64b) No.

set $\varepsilon = \frac{1}{2}$, $\forall N$,

set $x = 2^{-1/2N} < 1$

$f_n(x) \to 0$.

but $|f_{N+1}(x) - 0| = x^{N+1} \geq \left(\frac{1}{2^{1/2N}}\right)^{2N} = \frac{1}{2} = \varepsilon$

$\Rightarrow$ f_n is not converge uniformly on $[0, 1]$

Q2.64c) No. $\forall M > 0$

$\forall n > M$, $f_n(2) = 2^n = (1+1)^n > n \times 1 = M$

$\Rightarrow f_n(2)$ is not convergent.

Q2.64d) $\forall \varepsilon > 0$, $\forall x \in [0, 99/100]$

$\because 99/100 < 1$, $\exists N$ s.t. $(99/100)^N < \varepsilon$

$\forall n > N$, $|f_n(x) - 0| = x^n < (99/100)^n$

$< (99/100)^N < \varepsilon$

$\Rightarrow f_n$ uniformly convergent on $[0, 99/100]$.

2.66 a) $\forall \varepsilon > 0$, $\exists N$ where $\frac{1}{N} < \varepsilon$

$\forall x$, $\forall n > N$, $|f_n(x) - 0| = \left|\frac{\sin(n^2 x)}{n}\right| < \frac{1}{n} < \frac{1}{N} < \varepsilon$

$\Rightarrow f_n \to 0$ uniformly.

2.66 b) $f_n'(x) = n\cos(n^2 x)$

$f_n'(0) = n$ which is not a convergent seq.

$\therefore f_n'$ is not uniformly convergent.

2.68. $\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx = \frac{-n}{n+1}(1-x^2)^{n+1}\Big|_0^1 = \frac{n}{n+1} \to 1$

$\lim_{n\to\infty} f_n(x) = \begin{cases} \infty & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$

$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0 \neq \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

2.69. Weierstrass M test: $\forall n$, $\forall x \in S$

if $|f_n(x)| \leq M_n$ and $\sum_{i=1}^n M_i$ converge

then $a_n = \sum_{i=1}^n f_i(x)$ converge uniformly on S

pf : claim : cauchy seq is convergent.

suppose x_n is a cauchy seq.

$\forall \varepsilon > 0, \exists N \quad \forall n > N \quad \forall k$

$$|x_{n+k} - x_n| < \varepsilon/3$$

$\exists N_1$ s.t $\forall k \geq 1$

$$|x_{N_1+k} - x_{N_1}| < 1$$

$\Rightarrow \forall n, \quad |x_n| \leq \max\{|x_1|, |x_2|, \dots, |x_{N_1}|, |1 + x_{N_1}|, |x_{N_1} - 1|\}$

$\therefore$ x_n is a bounded seq.

$\therefore$ $a_n := \sup_{k \geq n} x_k$ exist.

a_n is montone decreasing

(see hw3)

$$a_n \leq a_1$$

$\therefore$ a_n convergent.

let $a_n \to a$,

$\exists N_2$ s.t $\forall n > N_2, |a_n - a| < \varepsilon/3$

$\forall n > \max\{N, N_2\}$

by def of a_n, $\exists x_m$ where $m \geq n$ s.t $x_n \leq a_n < x_m + \varepsilon/3$

$$0 \leq a_n - x_n \leq x_m - x_n + \varepsilon/3.$$

$$\begin{aligned}
|x_n - a| &\leq |x_n - a_n| + |a_n - a| \\
&\leq |x_m - x_n + \varepsilon/3| + \varepsilon/3 \\
&\leq |x_m - x_n| + \varepsilon/3 + \varepsilon/3 \\
&< \varepsilon
\end{aligned}$$

$\Rightarrow x_n \to a \quad \therefore$ cauchy seq is convergent.

let $g_n = \sum_{i=1}^{n} M_i \to g$.

$\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, $|g_n - g| < \varepsilon/2$

$\forall k \geq 1$, $\left|\sum_{i=n+1}^{n+k} M_i\right| = |g_{n+k} - g_n| \leq |g_{n+k} - g| + |g_n - g| < \varepsilon$

$$
\begin{aligned}
|a_{n+k} - a_n| &= \left|\sum_{i=n+1}^{n+k} f_i(x)\right| \\
&\leq \sum_{i=n+1}^{n+k} |f_i(x)| \\
&\leq \sum_{i=n+1}^{n+k} M_i = \left|\sum_{i=n+1}^{n+k} M_i\right| < \varepsilon
\end{aligned}
$$

$\therefore$ a_n is cauchy seq.

$\therefore$ a_n converge.