Doubling Properties of Self-similar Measures

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ABSTRACT. Let $\{F_i\}_{i=1}^N$ be a system of similitudes in $\mathbb{R}^n$. We study necessary and sufficient conditions for their associated self-similar measures to be doubling on its support. An equivalent condition is obtained when $\{F_i\}$ satisfies the open set condition. The condition allows us to construct many examples of interest. In the case where the open set condition is not satisfied, we study an infinitely convoluted Bernoulli measure (associated with the golden ratio $\rho = (\sqrt{5}-1)/2$) and give a necessary and sufficient condition for it to be doubling on its support $[0, 1]$.

1. INTRODUCTION

Let μ be a Borel measure supported on a subset K of $\mathbb{R}^n$. (Here and hereafter, for simplicity, 'Borel' will mean 'Borel-regular,' and we shall consider only finite non-negative measures.) We say that μ is doubling on K if there exists a constant $C > 0$ such that for any $x \in K$ and any $r > 0$, we have

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x));$$

here $B_r(x)$ denotes an (Euclidean) open ball centered at x and of radius r. Such measures arise naturally in harmonic analysis. For instance, the theory of Calderón-Zygmund singular integral operators can be developed when $\mathbb{R}^n$ is equipped with a Borel measure that is doubling on its support. The details can be found, for example, in Stein [13, Chapter 1]. Over there, only measures that are doubling on the whole $\mathbb{R}^n$ were considered. Nevertheless, the results readily generalize into the setting where the Borel measure μ is assumed only to be doubling on its support $K \subseteq \mathbb{R}^n$. It is also possible to develop a theory of Sobolev spaces assuming a Borel measure on $\mathbb{R}^n$ that is doubling on its support; see Hajłasz and Koskela [1]. In fact their approach in [1] is more general than is cited here; it applies to not only subsets of $\mathbb{R}^n$, but also to general metric spaces that are equipped with a doubling Borel measure.

In this paper we will mainly be concerned with necessary and sufficient conditions for a self-similar measure on $\mathbb{R}^n$ to be doubling on its support. Let $\{F_i\}_{i=1}^N$ be a finite system of similitudes on $\mathbb{R}^n$. Here by saying F_i is a similitude we mean that there exists $r_i \in (0,1)$ such that

$$|F_i(x) - F_i(y)| = r_i|x - y|$$

for all $x, y \in \mathbb{R}^n$, where $|\cdot|$ denotes the Euclidean norm. We call r_i the contraction ratio of F_i and the collection $\{F_i\}$ an *iterated function system* (IFS). Then as is well-known, there exists a unique non-empty compact subset $K \subseteq \mathbb{R}^n$ such that

$$K = \bigcup_{i=1}^{N} F_i(K),$$

which we call the attractor (or the self-similar set) of the IFS $\{F_i\}$; also, given any set of probability weights $\{p_i\}_{i=1}^N$ (which by definition satisfies $0 < p_i < 1$ for all i and $\sum_{i=1}^N p_i = 1$), there exists a unique Borel probability measure μ supported on K satisfying

$$\mu(A) = \sum_{i=1}^{N} p_i \mu(F_i^{-1}(A))$$

for all Borel subset A of $\mathbb{R}^n$. We call such μ a self-similar measure associated with $\{F_i\}$ with weights $\{p_i\}$, and we study necessary and sufficient conditions under which such μ is doubling on its support K.

Using the above notation, Olsen [10] showed that if

$$\Delta := \min\{d(K_i, K_j) \mid 1 \le i < j \le N\} > 0,$$

where K_i is a shorthand for $F_i(K)$ and $d(K_i, K_j)$ denotes the (Euclidean) distance between the compact sets K_i and K_j, then any self-similar measure μ associated with $\{F_i\}_{i=1}^N$ is doubling on its support K. In fact Olsen's theorem is more general. It works for graph-directed self-similar measures. See [10, Lemma 5.3]. The case for $\Delta = 0$ (for instance, connected self-similar sets) is more complicated, as we shall see. Mauldin and Urbański [8, Lemma 3.14] proved that if $\{F_i\}$ satisfies the *open set condition* (OSC), then the associated *canonical* self-similar measure is doubling on its support K. (The definitions are restated in Definitions 2.1 and 2.2.) Indeed their proof works for conformal measures associated with a finite conformal IFS, and they concluded that for such conformal measures m, there exists a constant $C > 0$ such that whenever x is in the 'attractor' of the conformal IFS and $r > 0$, we have

$$C^{-1} \le \frac{m(B_r(x))}{r^h} \le C,$$

with h being the Hausdorff dimension of the 'attractor.' In [9], they also proved a related result for infinite conformal IFS, providing a sufficient condition for the conformal measure to be doubling when the infinite conformal IFS is regular and satisfies a stronger separation condition than the OSC, namely the super-strong open set condition (SSOSC). They then used this as a tool to study infinite conformal IFS arising from continued fractions.

In what follows, in Section 2, we shall consider a finite system of similitudes $\{F_i\}$ on $\mathbb{R}^n$ that satisfies the OSC, together with its associated self-similar measures. We shall prove an equivalent condition on the weights of the self-similar measure for it to be doubling on its support:

Theorem 1.1. *Let $\{F_i\}_{i=1}^N$ be similitudes on $\mathbb{R}^n$ with contraction ratios $\{r_i\}_{i=1}^N$ that satisfy the OSC. Let K be its attractor, and let μ be a self-similar measure whose weights we denote by $\{p_i\}_{i=1}^N$. Then μ is doubling on K if and only if there exists a constant $C > 0$ such that for any (non-empty) finite words w and v that satisfy $K_w \subseteq \bar{B}(K_v, r_v)$, we have*

$$p_w \le C p_v.$$

Here $\bar{B}(F, r) := \{x \in \mathbb{R}^n \mid (x, F) \le r\}$ for a closed set F. See Theorem 2.3 for more details. As a corollary, we recover a special case of the result of Mauldin and Urbański which we quoted above:

Corollary 1.2. *Let $\{F_i\}_{i=1}^N$ and K be as in Theorem 1.1. Then its associated canonical self-similar measure μ is doubling on K.*

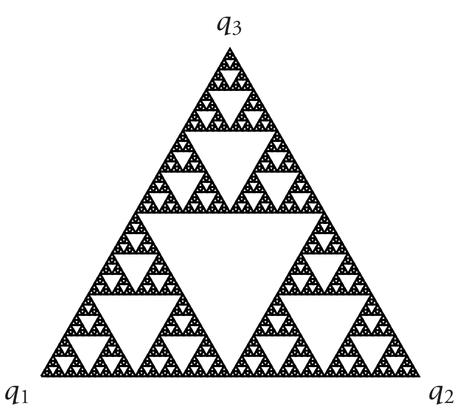
A number of interesting examples will be given in Section 3. We shall characterize, in these examples, the weights for which the self-similar measure is doubling on the attractor. To name a few, the first example represents a case where there is a severe restriction on the weights of a doubling self-similar measure:

Proposition 1.3. *Suppose that*

(a) $F_i(x) = (x + (i-1))/2$ $(i = 1, 2)$ *on $\mathbb{R}$ with attractor $[0, 1]$, or*

(b) q_1, q_2, q_3 *are the vertices of an equilateral triangle and* $F_i(z) = (z + q_i)/2$ $(i = 1, 2, 3)$ *on $\mathbb{R}^2$, in which case the attractor is the Sierpinski gasket SG.*

Then in both cases, a self-similar measure is doubling on the attractor if and only if it is the canonical one.

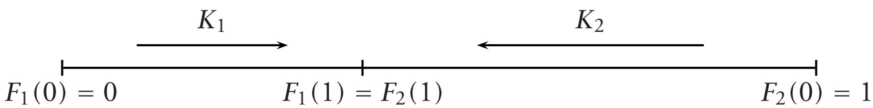


In contrast to the above, the second example represents a case where there is no restriction on the weights of a self-similar measure for it to be doubling on its support:

Proposition 1.4. *Let $\rho \in (0,1)$ and F_1, $F_2\colon [0,1] \to [0,1]$ be linear maps such that*

$$F_1(0) = 0, \quad F_1(1) = \rho = F_2(1) \quad \text{and} \quad F_2(0) = 1.$$

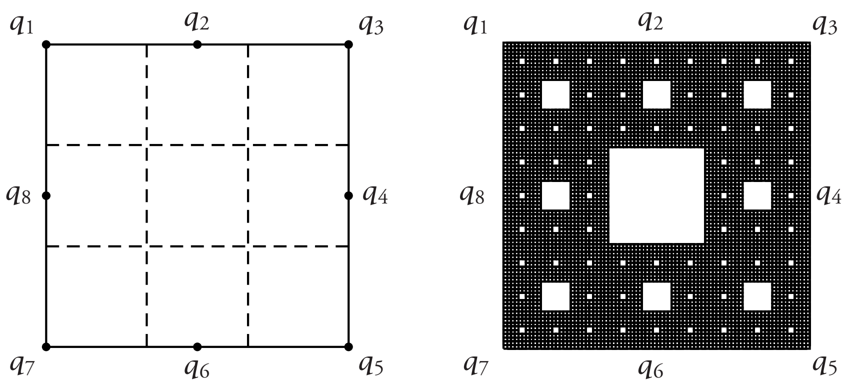
Then any self-similar measure μ associated with $\{F_1, F_2\}$ is doubling on its attractor $K = [0,1]$.



There are also intermediate situations, where there are some matching conditions on the weights for the self-similar measure to be doubling.

Proposition 1.5. *Let the Sierpinski carpet K be the attractor of $\{F_i\}_{i=1}^8$ on $\mathbb{R}^2$, where $F_i(z) = (z + q_i)/2$ for $i = 1, 2, \ldots, 8$ and $\{q_i\}_{i=1}^8$ are vertices and mid-points of the edges of a square as shown in the following figure. Then a self-similar measure μ (with weights $\{p_i\}_{i=1}^8$) is doubling on the carpet if and only if*

$$p_1 = p_3 = p_5 = p_7, \quad p_2 = p_6, \quad \text{and} \quad p_4 = p_8. \tag{1.1}$$



Some other interesting examples are also discussed in Section 3.

Finally, in Section 4, we shall consider the $[0,1]$ interval with a different self-similar structure. We shall consider it as the attractor of the system of similitudes $\{S_1, S_2\}$ on $\mathbb{R}$, where

$$S_1(x) = \rho x, \quad S_2(x) = \rho(x-1) + 1$$

and $\rho = (\sqrt{5}-1)/2$ is the golden ratio. This is interesting because $S_1[0,1]$ and $S_2[0,1]$ have intersection. In particular, this system of similitudes does not satisfy the open set condition. The study of self-similar measures associated with this system of similitudes is historically connected with probability theory: in fact the equal weight self-similar measure associated with $\{S_1, S_2\}$ is just an example of *infinitely convolved Bernoulli measures* (ICBM). (The reader is referred to [5, Section 1] for an interesting account of the history of ICBM. See also [12].) We shall prove the following theorem:

Theorem 1.6. *A self-similar measure μ associated with $\{S_1, S_2\}$ is doubling on $[0,1]$ if and only if its weights satisfy $p_1 = p_2 = \frac{1}{2}$.*

This will be done by using a special device of Strichartz [14]. (The same technique has been used in [6] to determine the L^q-spectrum of this equal-weight self-similar measure.)

It should be remarked that recently Kigami has independently discovered a set of equivalent conditions for the measure to be doubling while he was studying (upper and lower) heat kernel estimates on self-similar sets [4]. There he developed a more sophisticated language that is particularly suited to his purposes; he introduced the notion of *scales* on the symbolic space, which enables one to define on the self-similar set a one-parameter increasing family of open sets 'centered at a point' (that one can think of as a ball with a given center and radius), and a more general notion of doubling using these 'balls.' Our set-up is more direct and the conditions are easier to apply, and our target is more on the singular integral on self-similar sets in $\mathbb{R}^n$, rather than the heat kernel on the general metric-measure spaces. It is hoped that our more concrete approach will be more easily assimilated.

2. Open Set Condition and Doubling

Our main aim in this section is to prove the following necessary and sufficient condition for a self-similar measure to be doubling on its support. In the next section we shall illustrate the theorem with a number of interesting examples.

Definition 2.1. A family of similitudes $\{F_i\}_{i=1}^N$ on $\mathbb{R}^n$ is said to satisfy the *open set condition* (OSC) if there exists a non-empty bounded open set $O \subseteq \mathbb{R}^n$ such that

$$\bigcup_{i=1}^{N} F_i(O) \subseteq O$$

and

$$F_i(O) \cap F_j(O) = \varnothing$$

for all $i \neq j$, $1 \leq i, j \leq N$. We shall always denote by K the attractor of such $\{F_i\}_{i=1}^N$.

Let us remark here that if $\{F_i\}_{i=1}^N$ satisfies the OSC, then a self-similar measure μ associated with it satisfies $\mu(K_w) = p_w$ for all (finite) words w, where K_w

denotes $F_w(K)$ and $\{p_i\}_{i=1}^N$ are the weights of μ. Here we have adopted the common multi-index notation: if $w_1, w_2, \dots, w_m \in \{1, 2, \dots, N\}$, we call $w = w_1 w_2 \dots w_m$ a (finite) *word* of length m, and for such words we write

$$F_w := F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m},$$

and similarly $p_w := p_{w_1} p_{w_2} \dots p_{w_m}$, $r_w := r_{w_1} r_{w_2} \dots r_{w_m}$. These convenient notation shall be adopted throughout the paper.

Definition 2.2. Let $\{F_i\}_{i=1}^N$ on $\mathbb{R}^n$ be a family of similitudes that satisfies the open set condition, and let r_i be the contraction ratio for $1 \le i \le N$. Then if s is the (unique) solution to the equation

$$\sum_{i=1}^N r_i^s = 1,$$

and if μ is the self-similar measure associated with $\{F_i\}_{i=1}^N$ whose weight is $p_i = r_i^s$, we call μ the self-similar measure associated with $\{F_i\}_{i=1}^N$ with *natural weights*. We also call such μ the *canonical* self-similar measure associated with $\{F_i\}_{i=1}^N$.

Let us now prove the following theorem, which readily implies Theorem 1.1.

Theorem 2.3. *Let $\{F_i\}_{i=1}^N$ be similitudes on $\mathbb{R}^n$, with contraction ratios $\{r_i\}_{i=1}^N$, that satisfies the OSC. Let K be its attractor, and let μ be a self-similar measure whose weights we denote by $\{p_i\}_{i=1}^N$. Then the following are equivalent:*

(a) *μ is doubling on K;*

(b) *For any $C_1 > 0$, there exists $C_2 > 0$ such that for any (non-empty) finite words w and v that satisfy $K_w \subseteq \bar{B}(K_v, C_1 r_v)$, we have*

$$p_w \le C_2 p_v.$$

(c) *There exist constants C_1, $C_2 > 0$ such that for any (non-empty) finite words w and v that satisfy $K_w \subseteq \bar{B}(K_v, C_1 r_v)$, we have*

$$p_w \le C_2 p_v.$$

Proof. Rescaling if necessary, we assume that $\operatorname{diam}(K) = 1$. Obviously (b) implies (c). We shall first show that (c) implies (a), and then show that (a) implies (b).

(c) $\Rightarrow$ (a): Without loss of generality let us assume that (c) holds with some $C_1 < 2$. We define

$$r_{\max} = \max_{1 \le i \le N} r_i, \quad r_{\min} = \min_{1 \le i \le N} r_i, \quad \text{and} \quad p_{\min} = \min_{1 \le i \le N} p_i,$$

and we let k be the smallest positive integer for which $r_{\max}^{-k} > 4C_1^{-1}$. Also we take $\eta \in (0, r_{\min}^{k+1})$ and take

$$C_3 = \min\{\mu(K_w) \mid \operatorname{diam}(K_w) \geq \eta\} > 0.$$

Finally for $a \in (0,1)$ we let

$$\Lambda_a = \{w = w_1 w_2 \ldots w_m \mid r_{w_1 w_2 \ldots w_m} \leq a < r_{w_1 w_2 \ldots w_{m-1}}\},$$

and if further $x \in K$, we let

$$\Lambda_{a,x} = \{w \in \Lambda_a \mid d(x, K_w) \leq a\}.$$

It then follows (see, e.g., Kigami Proposition 1.5.8) that there exists $M \geq 1$ such that for all $a \in (0,1)$ and all $x \in K$,

$$\#\Lambda_{a,x} \leq M. \tag{2.1}$$

To prove μ is doubling on K, we will show that for any $x \in K$ and $r > 0$, we have

$$\mu(B_{2r}(x)) \leq \frac{MC_4}{p_{\min}^{k+1}} \mu(B_r(x)), \tag{2.2}$$

where $C_4 = \max\{C_2, C_3^{-1}\} \geq 1$:

Let $x \in K$ and $r > 0$ be given. Let π be the natural projection from the sequence space to K, and write $x = \pi(v_1 v_2 v_3 \ldots)$ for some infinite word $v_1 v_2 v_3 \ldots$. Take m to be the smallest positive integer such that $r_{v_1 v_2 \ldots v_m} < r$. Then $B_r(x) \supseteq K_{v_1 v_2 \ldots v_m}$, so

$$\mu(B_r(x)) \geq \mu(K_{v_1 v_2 \ldots v_m}) = p_{v_1 v_2 \ldots v_m}. \tag{2.3}$$

Now to prove (2.2) we want to estimate $\mu(B_{2r}(x))$; we look at two cases:

Case 1: $\operatorname{diam}(K_{v_1 v_2 \ldots v_m}) \geq \eta$.
It follows easily from the definition of C_3 that

$$\mu(B_{2r}(x)) \leq 1 \leq C_3^{-1} \mu(K_{v_1 v_2 \ldots v_m}) \leq C_3^{-1} \mu(B_r(x)) \leq \frac{MC_4}{p_{\min}^{k+1}} \mu(B_r(x)),$$

so (2.2) holds in this case.

Case 2: $\operatorname{diam}(K_{v_1 v_2 \ldots v_m}) < \eta$.
Then $r_{v_1 v_2 \ldots v_m} < \eta$, so $r_{\min}^m < \eta < r_{\min}^{k+1}$, from which it follows that $m > k+1$. Define v to be the finite word $v_1 v_2 \ldots v_{m-k-1}$. Then by minimality of m and the choice of k, we have

$$1 \geq r_v = \frac{r_{v_1 v_2 \ldots v_{m-1}}}{r_{v_{m-k} v_{m-k+1} \ldots v_{m-1}}} \geq \frac{r}{r_{\max}^k} > (2C_1^{-1})(2r).$$

So by $2r < (C_1/2) r_v < 1$, we have

$$B_{2r}(x) \cap K \subseteq \bigcup_{w \in \Lambda_{2r,x}} K_w, \tag{2.4}$$

and for each $w \in \Lambda_{2r,x}$, we have

$$\operatorname{diam}(K_w) \leq 2r < \frac{C_1}{2} r_v \tag{2.5}$$

with

$$d(K_w, K_v) \leq 2r < \frac{C_1}{2} r_v. \tag{2.6}$$

It follows that for each such w, we have

$$K_w \subseteq \bar{B}(K_v, C_1 r_v), \tag{2.7}$$

and we can then use our assumption in (c) to conclude that

$$p_w \leq C_4 p_v \tag{2.8}$$

holds for each $w \in \Lambda_{2r,x}$. Hence, by (2.1), (2.4) and (2.8), we have $\mu(B_{2r}(x)) \leq \sum_{w \in \Lambda_{2r,x}} p_w \leq M C_4 p_v$. Together with (2.3) it follows that in Case 2 we have

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} \leq \frac{M C_4 p_v}{p_{v_1 v_2 \ldots v_m}} = \frac{M C_4}{p_{v_{m-k} v_{m-k+1} \ldots v_m}} \leq \frac{M C_4}{p_{\min}^{k+1}},$$

so (2.2) is proved in Case 2 also, and we are done.

(a) $\Rightarrow$ (b): Let $C_1 > 0$ be arbitrary. Recall that $\{F_i\}$ satisfies the OSC; we let O be the open set in the definition of the OSC. Since we are working in $\mathbb{R}^n$, it is known (see, e.g. [2], [7] and [11]) that we may assume in addition that $O \cap K \neq \emptyset$. Thus if we take $x_0 \in O \cap K$ and $B_{R_0}(x_0) \subseteq O$, then choosing $\alpha^{-1} \in (0, R_0)$, we have, for any finite word v, that there exists $x \in K_v$ and $r > \alpha^{-1} r_v$ such that $B_r(x) \cap K \subseteq K_v$. (One can simply take $x = F_v(x_0)$ and

$r = r_v R_0$; then $B_r(x)$, being an open ball that is disjoint from all the open sets $F_\nu(O)$ for which ν does not begin with v, must be disjoint from all such K_ν as well, since $\overline{F_\nu(O)} \supseteq K_\nu$. This says $B_r(x) \cap K \subseteq K_v$.) Suppose now that μ is doubling on K. Then for the α chosen above, there exists a constant $C_2 > 0$ such that $\mu(B_{(C_1+1)\alpha r}(x)) \leq C_2\mu(B_r(x))$ for all $x \in K$ and all $r > 0$. So let w and v be finite words that satisfy

$$K_w \subseteq \bar{B}(K_v, C_1 r_v). \tag{2.9}$$

Then choose $x \in K_v$ and $r > \alpha^{-1}\operatorname{diam}(K_v)$ such that $B_r(x) \cap K \subseteq K_v$; we have, by (2.9), that

$$K_w \subseteq \bar{B}_{(C_1+1)r_v}(x) \subseteq B_{(C_1+1)\alpha r}(x).$$

It follows that

$$p_w = \mu(K_w) \leq \mu(B_{(C_1+1)\alpha r}(x)) \leq C_2\mu(B_r(x)) \leq C_2\mu(K_v) = C_2 p_v$$

and our assertion (b) is proved. ◻

Remark. In practice, to use part (c) of the above theorem to check that a self-similar measure μ is doubling on its support, we can assume in addition that the first letters of the words w and v satisfy

$$\begin{cases} w_1 \neq v_1, \\ K_{w_1} \cap K_{v_1} \neq \varnothing. \end{cases} \tag{2.10}$$

In other words, under the assumptions of Theorem 2.3, to show that μ is doubling on K, it suffices to verify the following condition:

(d) There exist constants $C_1, C_2 > 0$ such that for any (non-empty) finite words w and v that satisfy $w_1 \neq v_1$, $K_{w_1} \cap K_{v_1} \neq \varnothing$ and $K_w \subseteq \bar{B}(K_v, C_1 r_v)$, we have
$$p_w \leq C_2 p_v.$$

This is proved in the following:

Proof. (d) ⇒ (a): We use the notation in the above proof, except now we choose η to further satisfy

$$\eta < \min\{d(K_i, K_j) \mid K_i \cap K_j = \varnothing,\ 1 \leq i, j \leq N\}.$$

Then when $x \in K$ and $r > 0$ are given, we shall choose the v_i and m as before, so that the lower estimate (2.3) remains valid. To obtain an upper estimate for $\mu(B_{2r}(x))$, we again consider two cases: the proof in the case where $\operatorname{diam}(K_{v_1 v_2 \ldots v_m}) \geq \eta$ carries over, while in the case where $\operatorname{diam}(K_{v_1 v_2 \ldots v_m}) < \eta$, we still obtain (2.4), with (2.5), (2.6) and (2.7) all continuing to hold for all

$w \in \Lambda_{2r,x}$. However, since now we only assume (2.8) to hold for those pairs of words w and v for which both (2.7) and (2.10) are satisfied, we have to reduce the situation to the case where (2.10) also holds. Indeed for $w \in \Lambda_{2r,x}$, we see from (2.5) that the word v cannot begin with w, so either

(i) w begins with v, in which case obviously

$$p_w \le p_v \le C_4 p_v$$

so (2.8) still holds; or

(ii) there exists a positive integer ℓ such that $w_\ell \neq v_\ell$, and we assume this ℓ to be the smallest. We want to prove that p_w/p_v is bounded above by C_4. Without loss of generality we assume $\ell = 1$. (Otherwise consider the words w^* and v^* that are obtained from w and v, respectively, by removing their first $\ell - 1$ letters; then $p_w/p_v = p_{w^*}/p_{v^*}$, and the following argument works for w^* and v^* in place of w and v.) Again we consider two cases.

If $\operatorname{diam}(K_v) \ge \eta$, then p_w/p_v is at most

$$\frac{1}{\mu(K_v)} \le C_3^{-1} \le C_4,$$

so (2.8) holds.

If $\operatorname{diam}(K_v) < \eta$, then $r_v < \eta$, and from (2.6) we have

$$d(K_w, K_v) < \frac{C_1}{2} r_v < r_v < \eta$$

(recall without loss of generality we assumed $C_1 < 2$), so in this case by our additional assumption on η, we have $K_{w_1} \cap K_{v_1} \neq \emptyset$, and together with (2.7) and $w_1 \neq v_1$, we can invoke the condition (d) to conclude that (2.8) holds.

This proves that (2.8) holds in both cases. Then the upper estimate of $\mu(B_{2r}(x))$ follows as before, and so does the fact that μ is doubling on K. This completes the proof. ◻

We remark here that in applying the above theorem and corollary, only words w and v that are sufficiently long need to be considered, as should be apparent from the proof.

As a simple corollary of Theorem 1.1, we now prove Corollary 1.2, which is actually a special case of [8, Lemma 3.14].

Proof of Corollary 1.2. Let $\{r_i\}$ be the contraction ratios of $\{F_i\}$. Then the canonical self-similar measure μ associated with $\{F_i\}$ has weights $p_i = r_i^\alpha$, where $\alpha > 0$ is the number that satisfies $\sum_{i=1}^N r_i^\alpha = 1$. In view of Theorem 1.1, suppose that w and v are finite words that satisfy $K_w \subseteq \bar{B}(K_v, r_v)$. Then $r_w \le 3r_v$, so $p_w = r_w^\alpha \le 3^\alpha r_v^\alpha = 3^\alpha p_v$, and the equivalent condition of Theorem 1.1 holds with $C = 3^\alpha$. Hence such μ must be doubling on K. ◻

3. Examples of Doubling with OSC

In this section we go to the examples. With Corollary 1.2, we can now prove our characterization of the doubling self-similar measures on $[0,1]$ and SG, as given in Propostion 1.3.

Proof of Proposition 1.3. We already know, from Corollary 1.2, that the canonical self-similar measures on $[0,1]$ and SG are doubling on them respectively. Hence we only need to prove the converse.

First, on $[0,1]$, consider the words $w = 12^k$ and $v = 21^k$. By Theorem 1.1, we see that for a self-similar measure μ (whose weights we write as $\{p_i\}$) to be doubling on $[0,1]$, we must have the existence a constant $C > 0$ such that

$$\frac{p_1 p_2^k}{p_2 p_1^k} \leq C$$

holds for any positive integer k. Interchanging the roles of w and v, we indeed get the existence of a constant $C > 0$ such that

$$C^{-1} \leq \frac{p_1 p_2^k}{p_2 p_1^k} \leq C$$

for all positive integers k. This implies $p_1 = p_2$, so μ has to be the canonical self-similar measure.

The same assertion can also be proved by the following direct argument: suppose that the weights of the self-similar measure μ satisfy $p_2 < p_1$. Then letting

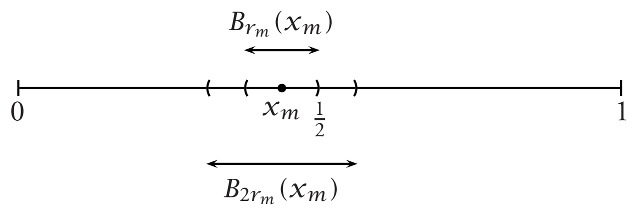
$$x_m = F_1 F_2^{m-1}(0) = \frac{1}{2} - 2^{-m} \in [0,1]$$

and $r_m = 2^{-m}$, we have

$$\begin{aligned}
\frac{\mu(B_{2r_m}(x_m))}{\mu(B_{r_m}(x_m))} &\geq \frac{\mu([\frac{1}{2}, \frac{1}{2}+2^{-m}])}{\mu([\frac{1}{2}-2^{1-m}, \frac{1}{2}])} \\
&= \frac{\mu(F_2 F_1^{m-1}[0,1])}{\mu(F_1 F_2^{m-2}[0,1])} \\
&= \frac{p_2 p_1^{m-1}}{p_1 p_2^{m-2}} \\
&\to \infty \quad \text{as } m \to \infty.
\end{aligned}$$

It follows that such μ can never be doubling on $[0,1]$. This completes our proof in the case of $[0,1]$.

An argument similar to the above works for SG. ◻



The key point here is that each piece of the self-similar set touches another piece essentially at a 'junction point.' This forces the non-canonical self-similar measures to fail to be doubling in the above examples.

The example above is a case where there are severe restrictions on the weights of a self-similar measure for it to be doubling on its support. Next we prove Proposition 1.4, which represents a case where there is no restriction on the weights of a self-similar measure for it to be doubling on the attractor, and still $\Delta = 0$ (here $\Delta := \min\{d(K_i, K_j) \mid 1 \le i < j \le N\}$ as in Section 1; recall that when $\Delta > 0$ any self-similar measure is doubling on the attractor). This may be contrasted with Proposition 1.3.

Proof of Proposition 1.4. According to Theorem 2.3 and the remark after it, to check whether a self-similar measure μ is doubling on $K = [0, 1]$, we only need to consider the pairs of cells K_{121^k} and K_{221^k}, where $k \in \mathbb{N}$. This is because if v and w are words that satisfy $K_w \subset \bar{B}(K_v, r_v)$ and $w_1 \neq v_1$, then w has length not much shorter than that of v (if not longer), so without loss of generality, we may assume that $v = 121^k$ and $w = 221^k$, or vice versa (observe that it suffices to consider the shortest possible w, since p_w decreases when the length of w increases). However, the measures of K_{121^k} and K_{221^k} match up automatically: indeed if we write the weights of μ as p_1 and p_2, then

$$\frac{\mu(K_{121^k})}{\mu(K_{221^k})} = \frac{p_1}{p_2},$$

which is independent of k. Thus for this self-similar structure, any self-similar measure on $[0, 1]$ is doubling on $[0, 1]$. $\square$

The case for the Sierpinski carpet is more interesting; the intersection of two different pieces is a line segment, and we show that there are doubling self-similar measures on the carpet that are not canonical, as was indicated in Proposition 1.5.

Proof of Proposition 1.5. The necessity is easy as always: let μ be a self-similar measure with weights $\{p_i\}_{i=1}^8$ that is doubling on the carpet K. Then by Theorem 1.1, considering the pairs of words $(15^k, 27^k)$ (note $F_1(q_5) = F_2(q_7)$, where q_5 and q_7 are fixed points of F_5 and F_7 respectively), we see that there is a constant

$C > 0$ such that

$$C^{-1} \leq \frac{p_1 p_5^k}{p_2 p_7^k} \leq C$$

for all $k \in \mathbb{N}$, which forces $p_5 = p_7$. Similarly, by considering the pairs of words $(15^k, 83^k)$, $(37^k, 41^k)$, $(16^k, 82^k)$, and $(14^k, 28^k)$, where $k \in \mathbb{N}$, we see that we must have (1.1) holding if μ is to be doubling on K. This proves the necessity.

Next, the proof of the sufficiency follows from Theorem 1.1. Suppose that the weights of the self-similar measure μ satisfy (1.1). Let w and v be finite words that satisfy $w_1 \neq v_1$, $K_{w_1} \cap K_{v_1} \neq \emptyset$ and $K_w \subseteq \bar{B}(K_v, r_v)$. Then writing $w = w_1 w_2 \ldots w_s$ and $v = v_1 v_2 \ldots v_m$, we have $s \geq m$. Let us introduce an equivalent relation $\sim$ by

$$1 \sim 3 \sim 5 \sim 7, \quad 2 \sim 6 \quad \text{and} \quad 4 \sim 8.$$

Since $d(K_w, K_v) < r_v$ = the size of a level m cell, a simple consideration of the geometry of the carpet shows that either

$$w_i \sim v_i \quad \text{for all } 1 \leq i \leq m,$$

or there exists $1 \leq i_0 \leq m$ such that

$$\begin{cases} w_i \sim v_i & \text{for all } i < i_0, \\ w_{i_0} \not\sim v_{i_0}, & \\ w_i \sim 1 \sim v_i & \text{for all } i_0 < i \leq m. \end{cases}$$

In either case, since $p_i = p_j$ whenever $i \sim j$, we have

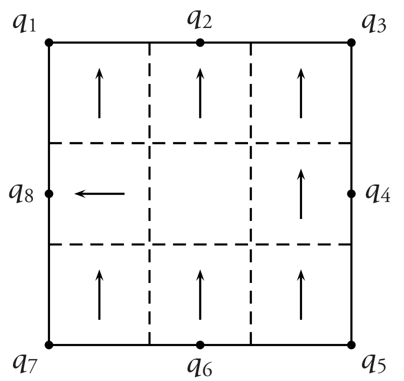
$$\frac{p_w}{p_v} \leq \frac{1}{p_{\min}}.$$

Hence in view of Theorem 1.1, μ must be doubling on K. $\square$

With the exception of Proposition 1.4, the similitudes so far are only translates of contractions towards the origin. Below we consider similitudes that involve rotations:

Proposition 3.1. *Let $\{q_i\}_{i=1}^8$ and $\{F_i\}_{i=1}^8$ be as in Proposition 1.5. Suppose that $\tilde{F}_i = F_i$ for $i = 1, 2, \ldots, 7$, and let $\tilde{F}_8 = F_8 \circ R$ where R is a counter-clockwise rotation through an angle of $\pi/2$ about the origin. Then the attractor K is still the Sierpinski carpet, and a self-similar measure $\tilde{\mu} = \sum_{i=1}^8 \tilde{p}_i \tilde{\mu} \circ \tilde{F}_i^{-1}$ is doubling on K if and only if*

$$\tilde{p}_1 = \tilde{p}_3 = \tilde{p}_5 = \tilde{p}_7 \quad \text{and} \quad \tilde{p}_2 = \tilde{p}_4 = \tilde{p}_6. \tag{3.1}$$



Proof. In fact $\tilde{p}_2 = \tilde{p}_4 = \tilde{p}_6$ is necessary for $\tilde{\mu}$ to be doubling on K, because $\tilde{F}_8(q_4) = \tilde{F}_1(q_6)$ and $\tilde{F}_1(q_4) = \tilde{F}_2\tilde{F}_8(q_2)$. Clearly we also need $\tilde{p}_1 = \tilde{p}_3 = \tilde{p}_5 = \tilde{p}_7$ for μ to be doubling on K, as in Proposition 1.5. This proves the necessity of (3.1) for μ to be doubling on K.

The proof of the converse implication is more complicated. It depends on the following fact: If $K_w := \tilde{F}_w(K)$ is a cell that intersects the straight line segment joining q_1 and q_3, then the following eight cells have measures all comparable to one another:

$$\begin{array}{cccc} K_w, & R(K_w), & R^2(K_w), & R^3(K_w), \\ -K_w, & -R(K_w), & -R^2(K_w), & -R^3(K_w). \end{array}$$

(Here the R is as in the rotation as in the statement of the proposition, R^2 denotes the composition of two R, and $-K_w$ denotes the set of all $z \in \mathbb{R}^2$ such that $-z \in K_w$, etc.) In fact if we take any two cells from the above eight cells, then the ratio of their measures must be equal to 1, p_2/p_8 or p_8/p_2, which can be proved by induction on the length of w. Granting this, a careful analysis of the geometry of the carpet (similar to the one in the proof of Proposition 1.5) shows that whenever w and v are two finite words for which $w_1 \neq v_1$, $K_{w_1} \cap K_{v_1} \neq \emptyset$ and $K_w \subseteq \bar{B}(K_v, r_v)$, then

$$\frac{\tilde{p}_w}{\tilde{p}_v} \leq \frac{1}{\tilde{p}_{\min}^2},$$

where $\tilde{p}_{\min} = \min_{1 \leq i \leq 8} \tilde{p}_i$. This proves the sufficiency of (3.1) for $\tilde{\mu}$ to be doubling on K. $\square$

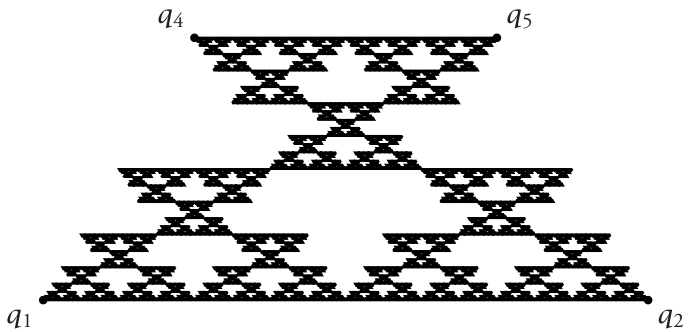
This is an example where the similitudes involve reflections:

Proposition 3.2. *Let q_1, q_2, q_3 be the vertices of an equilateral triangle. Let $F_i(z) = (z + q_i)/2$ for $i = 1, 2$, and let*

$$F_3(z) = R\left(\frac{z + q_3}{2}\right)$$

where R is the reflection about the line joining $q_4 := (q_1 + q_3)/2$ and $q_5 := (q_2 + q_3)/2$. If μ is an associated self-similar measure whose weights we write as $\{p_i\}_{i=1}^3$, then it is doubling on the attractor K of $\{F_i\}$ if and only if $p_1 = p_2$.

The attractor K is a connected set as in the following figure:



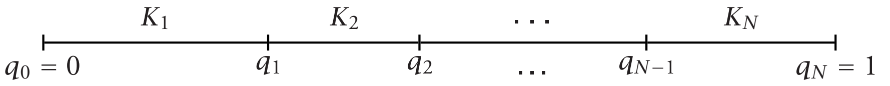
Proof. Observe that F_1, F_2, F_3 all map the trapezium $q_4q_1q_2q_5$ into itself, and they satisfy the open set condition with the open set being the interior of the trapezium. For μ to be doubling on the attractor K, according to Theorem 2.3 and the remark after it, we only need to match up the measures of the following pairs of cells (which are of the same size and intersect along $\bigcup_{i\neq j} K_i \cap K_j$):

$$(K_{231^m}, K_{332^m}), \quad (K_{132^m}, K_{331^m}) \quad \text{and} \quad (K_{12^m}, K_{21^m}).$$

This can be achieved if and only if $p_1 = p_2$. $\square$

Again with the exception of Proposition 1.4, the examples so far involve only similitudes of equal contraction ratios. It is indeed possible, and not too difficult, to treat the case where the similitudes have different contraction ratios. The following is the simplest example:

Proposition 3.3. *Let $0 = q_0 < q_1 < q_2 < \cdots < q_N = 1$ be a partition of $[0, 1]$ ($N \geq 2$), and let $\{F_i\}_{i=1}^N$ be a family of linear maps that satisfies $F_i(0) = q_{i-1}$ and $F_i(1) = q_i$. Denote the contraction ratios of each F_i by r_i. Then the associated self-similar measure μ is doubling on $K := [0, 1]$ if and only if there exists $\alpha > 0$ such that $p_1 = r_1^\alpha$ and $p_N = r_N^\alpha$; here $\{p_i\}_{i=1}^N$ are the weights of μ.*



Proof. A simple consideration of the geometry, together with Theorem 2.3 and the remark after it, shows that a self-similar measure μ is doubling on $K := [0,1]$ if and only if there exists $C > 0$ such that the following holds for any $1 \le i \le N-1$:

$$\begin{cases} \operatorname{diam}(K_{iN^m}) \le \operatorname{diam}(K_{(i+1)1^k}) & \Rightarrow \quad \mu(K_{iN^m}) \le C\mu(K_{(i+1)1^k}), \\ \operatorname{diam}(K_{(i+1)1^k}) \le \operatorname{diam}(K_{iN^m}) & \Rightarrow \quad \mu(K_{(i+1)1^k}) \le C\mu(K_{iN^m}). \end{cases}$$

Simply put, this is saying that

$$\left\{(m,k) \in \mathbb{N}^2 \mid r_i r_N^m \le r_{i+1} r_1^k\right\} \subseteq \left\{(m,k) \in \mathbb{N}^2 \mid p_i p_N^m \le C p_{i+1} p_1^k\right\}$$

and

$$\left\{(m,k) \in \mathbb{N}^2 \mid r_{i+1} r_1^k \le r_i r_N^m\right\} \subseteq \left\{(m,k) \in \mathbb{N}^2 \mid p_{i+1} p_1^k \le C p_i p_N^m\right\}.$$

But here the p_i, p_{i+1}, r_i, and r_{i+1} are not important; indeed it is easy to show that there exists $C > 0$ such that the above holds for all $1 \le i \le N-1$ if and only if there exists $C_1 > 0$ such that

$$\left\{(m,k) \in \mathbb{N}^2 \mid r_N^m \le r_1^k\right\} \subseteq \left\{(m,k) \in \mathbb{N}^2 \mid p_N^m \le C_1 p_1^k\right\}$$

and

$$\left\{(m,k) \in \mathbb{N}^2 \mid r_1^k \le r_N^m\right\} \subseteq \left\{(m,k) \in \mathbb{N}^2 \mid p_1^k \le C_1 p_N^m\right\}.$$

As a result, μ is doubling on $[0,1]$ if and only if there exists $C_1 > 0$ such that for any m, $k \in \mathbb{N}$, we have

$$\left\{(m,k) \in \mathbb{N}^2 \mid m \ge \frac{\log r_1}{\log r_N} k\right\} \subseteq \left\{(m,k) \in \mathbb{N}^2 \mid m \ge \frac{\log p_1}{\log p_N} k + \frac{\log C_1}{\log p_N}\right\}$$

and

$$\left\{(m,k) \in \mathbb{N}^2 \mid k \ge \frac{\log r_N}{\log r_1} m\right\} \subseteq \left\{(m,k) \in \mathbb{N}^2 \mid k \ge \frac{\log p_N}{\log p_1} m + \frac{\log C_1}{\log p_1}\right\}.$$

This is equivalent to

$$\frac{\log p_1}{\log p_N} \ge \frac{\log r_1}{\log r_N} \quad \text{and} \quad \frac{\log p_N}{\log p_1} \ge \frac{\log r_N}{\log r_1},$$

i.e.,

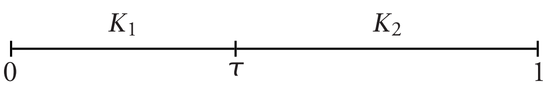
$$\frac{\log p_1}{\log r_1} = \frac{\log p_N}{\log r_N}.$$

If we call the above common ratio α (> 0), then $p_1 = r_1^\alpha$ and $p_N = r_N^\alpha$. This proves that μ is doubling on $[0, 1]$ if and only if there exists $\alpha > 0$ such that $p_1 = r_1^\alpha$ and $p_N = r_N^\alpha$. $\square$

As a special case, when there are only two similitudes, the above theorem reduces to the following result:

Corollary 3.4. *Let $\tau \in (0, 1)$ and F_1, F_2: $[0, 1] \to [0, 1]$ be defined by $F_1(x) = \tau x$, and $F_2(x) = (1 - \tau)x + \tau$. Then an associated self-similar measure μ is doubling on $[0, 1]$ if and only if there exists $\alpha > 0$ such that the weights $\{p_1, p_2\}$ of μ satisfies $p_1 = \tau^\alpha$ and $p_2 = (1 - \tau)^\alpha$; this happens if and only if μ is the Lebesgue measure on $[0, 1]$.*

In fact, since it is required that $p_1 + p_2 = 1$, when $p_1 = \tau^\alpha$ and $p_2 = (1-\tau)^\alpha$, we have $\tau^\alpha + (1 - \tau)^\alpha = 1$, so $\alpha = 1$, $p_1 = \tau$ and $p_2 = 1 - \tau$. It follows that μ is the usual Lebesgue measure on $K = [0, 1]$.



Finally, we sketch two more sophisticated examples of how Theorem 1.1 can be used to determine the doubling measures on a self-similar set.

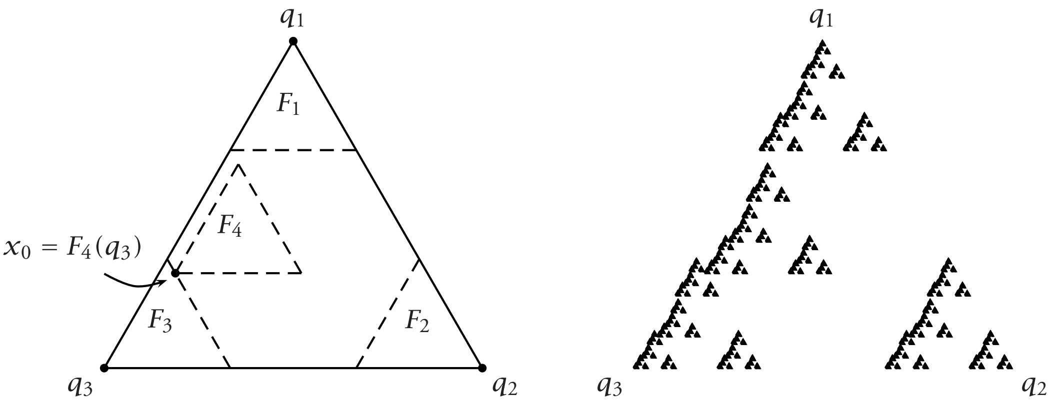
Proposition 3.5. *Let q_1, q_2, q_3 be the vertices of an equilateral triangle, and let F_i ($i = 1, 2, 3$) be defined by $F_i(z) = (z + q_i)/3$. Let F_4 be F_3 followed by a translation such that $F_4(q_3) = x_0$, where $x_0 = \lim_{k\to\infty} F_{w_k}(q_1)$ and*

$$w_k = 31212^3\,12^5 \cdots 12^{2k-1}$$

for all positive integers k. Then a self-similar measure $\mu = \sum_{i=1}^4 p_i \mu \circ F_i^{-1}$ is doubling on the attractor K if and only if $p_1 = p_2 = p_3$.

Proof. Observe that $\{F_i\}$ are similitudes that satisfies the OSC, with the open set being the interior of the triangle $q_1 q_2 q_3$. So the sufficiency is clear from Theorem 1.1: we only need to consider the cells that contain the point $x_0 = F_4(q_3)$. The proof of necessity is harder, and goes as follows:

In fact if a self-similar measure μ is to be doubling on K, then since $K_{43^{k^2+k}}$ and K_{w_k} are cells of the same size that have a non-empty intersection, by Theorem 1.1, there must exist $C > 0$ such that their μ-measures $p_4 p_3^{k^2+k}$ and $p_3 p_1^k p_2^{k^2}$ have



ratios bounded by C, i.e.,

$$C^{-1} \leq \frac{p_4 p_3^{k^2+k}}{p_3 p_1^k p_2^{k^2}} \leq C.$$

As a result, there exists a constant $C_1 > 0$ such that

$$C_1^{-1} \leq \frac{p_3^{k^2+k}}{p_1^k p_2^{k^2}} \leq C_1$$

for any $k \in \mathbb{N}$. Taking logarithm, we get

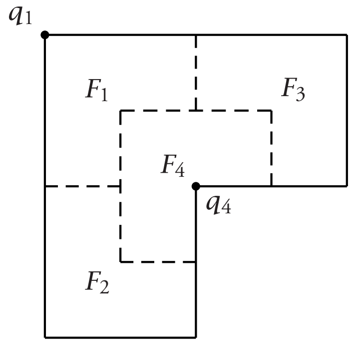
$$-\log C_1 \leq (\log p_3 - \log p_2)k^2 + (\log p_3 - \log p_1)k \leq \log C_1$$

for all $k \in \mathbb{N}$. This implies $\log p_3 - \log p_2 = \log p_3 - \log p_1 = 0$, so $p_1 = p_2 = p_3$. ∎

Proposition 3.6. *Let $q_1 = (-1, 1)$ and $q_4 = (0, 0)$. Let $F_1(z) = (z + q_1)/2$ and $F_4(z) = (z + q_4)/2$. Also let*

$$F_2 = R_{\pi/2} \circ F_1 \quad \text{and} \quad F_3 = R_{-\pi/2} \circ F_1,$$

where R_θ denotes the counter-clockwise rotation about the origin through an angle θ. Then letting K be the L-shaped region obtained by removing the square $(0, 1] \times [-1, 0)$ from $[-1, 1] \times [-1, 1]$, we see that K is the attractor of $\{F_i\}_{i=1}^4$, $\{F_i\}$ satisfies the open set condition with the open set being the interior of K, and a self-similar measure $\mu = \sum_{i=1}^4 p_i \mu \circ F_i^{-1}$ is doubling on K if and only if $p_1 = p_4$.



Proof. There are a lot of cells that we have to match up initially, but after applying the condition $p_1 = p_4$ (which is obviously necessary for μ to be doubling since $F_1(q_4) = F_4(q_1)$) to match up the cells that are close to the intersection of K_1 and K_4, it turns out that the rest all reduce to the matching of the measures of the cells K_w and $\tilde{R}(K_w)$, where K_w is a cell that intersects the line segment joining $(-1, 1)$ and $(-1, -1)$, and $\tilde{R}$ is the reflection along the line joining q_1 and q_4. The measures of these cells 'automatically' match up, because each such pairs of cells correspond to pairs of words that takes the form

$$1^{k_1}21^{k_2}31^{k_3}21^{k_4}3\ldots \quad \text{and} \quad 1^{k_1}31^{k_2}21^{k_3}31^{k_4}2\ldots,$$

where the k_i are non-negative integers. This finishes the sketch of the proof of that μ is doubling on K. ∎

4. Bernoulli Convolution and Golden Ratio

Let $\rho = (\sqrt{5}-1)/2$ be the golden ratio, and let $S_1(x) = \rho x$, $S_2(x) = \rho(x-1)+1$ be contractions. Then $[0, 1]$ is the attractor of $\{S_1, S_2\}$. Let

$$\mu = p_1\mu \circ S_1^{-1} + p_2\mu \circ S_2^{-1}$$

be the associated self-similar measure on $[0, 1]$, where $0 < p_1 \le p_2 < 1$ and $p_1 + p_2 = 1$. We shall prove Theorem 1.6 by showing that this μ is doubling on $[0, 1]$ if and only if $p_1 = p_2 = \frac{1}{2}$.

The theorem is interesting because $S_1[0, 1]$ and $S_2[0, 1]$ have large overlaps, which are in general difficult to handle. To overcome this difficulty, we make use of the clever observation due to Strichartz [14]:

Let $T_0 = S_1S_1$, $T_1 = S_1S_2S_2 = S_2S_1S_1$ and $T_2 = S_2S_2$. Then $\{T_0, T_1, T_2\}$ is a system of similitudes on $\mathbb{R}$, with $[0, 1]$ being its attractor. Moreover, $\{T_0, T_1, T_2\}$ satisfies the OSC, with the open set being $(0, 1)$. The only difficulty here is that the measure μ defined as above is not self-similar with respect to $\{T_0, T_1, T_2\}$, which prevents us from directly using the results in Section 2. However, as observed in [14], μ does satisfy a second-order self-similar identity with respect to $\{T_0, T_1, T_2\}$:

For any Borel set $X \subseteq [0,1]$ and for $i = 0, 1, 2$, we have

$$\begin{pmatrix} \mu(T_0T_iX) \\ \mu(T_1T_iX) \\ \mu(T_2T_iX) \end{pmatrix} = Q_i \begin{pmatrix} \mu(T_0X) \\ \mu(T_1X) \\ \mu(T_2X) \end{pmatrix}, \tag{4.1}$$

where

$$Q_0 = \begin{pmatrix} p_1^2 & 0 & 0 \\ p_1^2p_2 & p_1p_2 & 0 \\ 0 & p_2 & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & p_1^2 & 0 \\ 0 & p_1p_2 & 0 \\ 0 & p_2^2 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & p_1 & 0 \\ 0 & p_1p_2 & p_1p_2^2 \\ 0 & 0 & p_2^2 \end{pmatrix}.$$

Also μ is continuous, with

$$\begin{cases} \mu(T_0[0,1]) = \dfrac{p_1^2}{1-p_1p_2}, \\ \mu(T_1[0,1]) = \dfrac{p_1p_2}{1-p_1p_2}, \\ \mu(T_2[0,1]) = \dfrac{p_2^2}{1-p_1p_2}. \end{cases}$$

We will use the above to prove Theorem 1.6.

Proof of the necessity of Theorem 1.6. If on the contrary $0 < p_1 < p_2 < 1$, we shall show that the self-similar measure μ is not doubling on $[0,1]$:

Clearly, for any non-negative integers m, $T_1T_2^m[0,1]$ and $T_2T_0^{m+1}[0,1]$ are two intervals that intersect only at a point, and we have

$$\frac{|T_1T_2^m[0,1]|}{|T_2T_0^{m+1}[0,1]|} = \frac{\rho^{2m+3}}{\rho^{2m+4}} = \frac{1}{\rho}.$$

We will compute $\mu(T_1T_2^m[0,1])$ and $\mu(T_2T_0^m[0,1])$. First, observe that by (4.1), we have, inductively, that for any $m > 0$,

$$\mu(T_2^m[0,1]) = p_2^2\mu(T_2^{m-1}[0,1]) = \cdots = p_2^{2m-2}\mu(T_2[0,1]) = \frac{p_2^{2m}}{1-p_1p_2}.$$

As a result, we can prove inductively that for $m \geq 0$, we have

$$\mu(T_1 T_2^m[0,1]) = \frac{p_1 p_2^{m+1}}{1 - p_1 p_2} \sum_{j=0}^{m+1} p_1^j p_2^{m+1-j}. \tag{4.2}$$

Indeed, the case $m = 0$ is trivial, and when the above holds for some non-negative integer $m - 1$, we have, by (4.1) again, that

$$\begin{aligned}
\mu(T_1 T_2^m[0,1]) &= p_1 p_2 \mu(T_1 T_2^{m-1}[0,1]) + p_1 p_2^2 \mu(T_2^m[0,1]) \\
&= p_1 p_2 \frac{p_1 p_2^m}{1 - p_1 p_2} \sum_{j=0}^{m} p_1^{m-j} p_2^j + p_1 p_2^2 \frac{p_2^{2m}}{1 - p_1 p_2} \\
&= \frac{p_1 p_2^{m+1}}{1 - p_1 p_2} \sum_{j=0}^{m+1} p_1^{m+1-j} p_2^j.
\end{aligned}$$

This proves (4.2), and by symmetry, we have

$$\mu(T_1 T_0^m[0,1]) = \frac{p_1^{m+1} p_2}{1 - p_1 p_2} \sum_{j=0}^{m+1} p_1^j p_2^{m+1-j}, \quad m \geq 0.$$

Hence from (4.1) we see that

$$\begin{aligned}
\mu(T_2 T_0^{m+1}[0,1]) &= p_2 \mu(T_1 T_0^m[0,1]) \\
&= \frac{p_1^{m+1} p_2^2}{1 - p_1 p_2} \sum_{j=0}^{m+1} p_1^j p_2^{m+1-j}, \quad m \geq 0.
\end{aligned}$$

It follows that

$$\frac{\mu(T_1 T_2^m[0,1])}{\mu(T_2 T_0^{m+1}[0,1])} = \frac{1}{p_2} \left(\frac{p_2}{p_1} \right)^m$$

for all non-negative integers m. Since $0 < p_1 < p_2 < 1$, we see that the above fraction tends to infinity as $m \to \infty$. This proves that μ is not doubling on $[0,1]$. $\square$

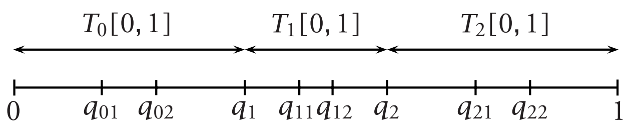
Next we prove the sufficiency of Theorem 1.6. Suppose that $p_1 = p_2 = \frac{1}{2}$, i.e.,

$$\mu = \frac{1}{2} \mu \circ S_1^{-1} + \frac{1}{2} \mu \circ S_2^{-1}. \tag{4.3}$$

Then the Q_i in the self-similar identities (4.1) become

$$Q_0 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \tag{4.4}$$

with $\mu(T_0[0,1]) = \mu(T_1[0,1]) = \mu(T_2[0,1]) = \frac{1}{3}$. For later convenience, let us write $q_1 = T_0(1)$ $(= T_1(0) = \rho^2)$, $q_2 = T_1(1)$ $(= T_2(0) = 1 - \rho^2 = \rho)$, and $q_{ij} = T_i(q_j)$ for $i = 0, 1, 2$ and $j = 1, 2$. We will call these 8 points 'junction points.'



To show that μ is doubling on $[0,1]$, we will use the following lemma:

Lemma 4.1. *There is a constant M such that whenever $x \in [0,1]$ and $r > 0$ satisfy $q_1 \in B_{2r}(x)$, we have $\mu(B_{2r}(x)) \leq M\mu(B_r(x))$.*

Proof. As in the proof of Theorem 1.1, without loss of generality, we will only consider the case where r is sufficiently small, for if we restrict to large r, then $B_r(x)$ will always contain a cell K_w that is not too small, so

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} \leq \frac{1}{\mu(K_w)}$$

is always bounded by a constant that is independent of x and r.

First, since $B_{2r}(x) \subseteq B_{4r}(q_1)$, if we take m to be the largest positive integer such that $4r \leq \rho^{2m+3}$ (such m exists since r is assumed to be small), then

$$B_{2r}(x) \subseteq T_0T_2^m[0,1] \cup T_1T_0^m[0,1]$$

since the set on the right hand side contains $B_{\rho^{2m+3}}(q_1) \supseteq B_{4r}(q_1)$. Note that by a direct computation using (4.1), we have

$$\mu(T_0T_2^m[0,1]) = \frac{1}{3}\left(0 + \frac{2}{4^m} + \frac{m-1}{4^m}\right) = \frac{m+1}{3 \cdot 4^m}$$

and

$$\mu(T_1T_0^m[0,1]) = \frac{1}{3}\left(\frac{m}{2 \cdot 4^m} + \frac{1}{4^m} + 0\right) = \frac{m+2}{6 \cdot 4^m}.$$

Hence we obtain an upper estimate for $\mu(B_{2r}(x))$, namely

$$\mu(B_{2r}(x)) \leq \mu(T_0T_2^m[0,1]) + \mu(T_1T_0^m[0,1]) \leq \alpha\frac{m}{4^m}$$

for some universal constant $\alpha > 0$.

Next, since by maximality of m, $\rho^{2(m+1)+3} < 4r$, we have $\rho^{2(m+4)} < r$. But

$$\rho^{2(m+4)} = \min|T_w[0,1]|,$$

where $|\cdot|$ denotes the Euclidean length of an interval and the minimum is taken over all words of length $m+4$. Since $B_r(x)$ is an interval contained in $T_0T_2^m[0,1] \cup T_1T_0^m[0,1]$, we infer that $B_r(x)$ contains an interval of the form $T_0T_2^mT_iT_jT_k[0,1]$ or $T_1T_0^mT_iT_jT_k[0,1]$ for some i, j, $k \in \{0,1,2\}$. It follows that $\mu(B_r(x)) \geq \min\{c_1, c_2\}$, where

$$\begin{cases} c_1 = \min\limits_{i,j,k\in\{0,1,2\}} \mu(T_0T_2^mT_iT_jT_k[0,1]), \\ c_2 = \min\limits_{i,j,k\in\{0,1,2\}} \mu(T_1T_0^mT_iT_jT_k[0,1]). \end{cases}$$

To estimate c_2, note that for i, j, $k \in \{0,1,2\}$, we have, by (4.1), that

$$\begin{aligned} &\mu(T_1T_0^mT_iT_jT_k[0,1]) \\ &\qquad = \frac{1}{3} \text{ sum of all entries in the second row of } Q_0^mQ_iQ_jQ_k. \end{aligned}$$

However, if a matrix R has non-negative entries in the second row and has an entry in the second row that is at least γ, then for $t = 0, 1, 2$, we have that RQ_t has an entry in its second row that is at least $\frac{1}{4}\gamma$. (This uses the fact that each row of Q_t has an entry $\geq \frac{1}{4}$ and that each Q_t has only non-negative entries; recall that now the Q_t are given by (4.4).) Successively apply the fact above to

$$\begin{cases} R = Q_0^m \\ t = i \end{cases}, \quad \begin{cases} R = Q_0^mQ_i \\ t = j \end{cases}, \quad \text{and} \quad \begin{cases} R = Q_0^mQ_iQ_j \\ t = k \end{cases},$$

and using the fact that Q_0^m has an entry of $(m/2)(1/4^m)$ in its second row, we get that $Q_0^mQ_iQ_jQ_k$ has an entry of size at least $(1/4^3)(m/2)(1/4^m)$ in its second row. Hence

$$\mu(T_1T_0^mT_iT_jT_k[0,1]) \geq \beta\frac{m}{4^m}$$

where β is a universal constant. This holds for all i, j, $k \in \{0,1,2\}$, so

$$c_2 \geq \beta\frac{m}{4^m}.$$

Similarly, noting that the first row of Q_2^m is $(0,\ 2/4^m,\ (m-1)/4^m)$, we have that Q_2^m always contains an entry of size $\max\{2/4^m,\ (m-1)/4^m\} \geq (m/2)(1/4^m)$ in its first row (note now $m \geq 1$), so by the same argument as above we get

$$c_1 \geq \beta \frac{m}{4^m}.$$

As a result, we get our desired lower bound for $\mu(B_r(x))$, namely that

$$\mu(B_r(x)) \geq \min\{c_1, c_2\} \geq \beta \frac{m}{4^m}.$$

Together with our estimate for $\mu(B_{2r}(x))$, we get

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x)))} \leq \frac{\alpha m/4^m}{\beta m/4^m} = M$$

for some constant M independent of x and r. □

Proof of the sufficiency of Theorem 1.6. Let $x \in [0,1]$ and $r > 0$ be given. We shall show that

$$\mu(B_{2r}(x)) \leq M\mu(B_r(x)) \tag{4.5}$$

holds for the same M as in Lemma 4.1. Note that this is trivial if $q_1 \in B_{2r}(x)$; by symmetry, this also holds if $q_2 \in B_{2r}(x)$. Now suppose that $q_{02} \in B_{2r}(x)$ but $q_1 \notin B_{2r}(x)$; then by Lemma 4.1 applied to the concentric balls $S_1^{-1}(B_r(x)) \subseteq S_1^{-1}(B_{2r}(x)) \ni q_1$, we get, by (4.3), that

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} = \frac{\frac{1}{2}\mu(S_1^{-1}(B_{2r}(x)))}{\frac{1}{2}\mu(S_1^{-1}(B_r(x)))} \leq M$$

so (4.5) holds. Repeating this argument, we see that (4.5) also holds if $q_{01} \in B_{2r}(x)$ but $q_{02} \notin B_{2r}(x)$. Hence (4.5) holds once $B_{2r}(x)$ contains q_{01} or q_{02}. By symmetry, the same is true if $B_{2r}(x)$ contains q_{21} or q_{22}. Furthermore, if $q_{11} \in B_{2r}(x)$, then the above argument gives

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} = \frac{\frac{1}{2}\mu(S_1^{-1}(B_{2r}(x))) + \frac{1}{2}\mu(S_2^{-1}(B_{2r}(x)))}{\frac{1}{2}\mu(S_1^{-1}(B_r(x))) + \frac{1}{2}\mu(S_2^{-1}(B_r(x)))} \leq M$$

since $S_1^{-1}(B_{2r}(x))$ contains q_{01} and $S_2^{-1}(B_{2r}(x))$ contains q_{21}. As a result, (4.5) holds in this case as well, and by symmetry the same holds if $q_{12} \in B_{2r}(x)$. Thus we have proven that (4.5) holds once $B_{2r}(x)$ contains one of the 8 'junction points.'

Now suppose that $B_{2r}(x)$ does not contain any of the 8 'junction points.' Then it is contained in some $T_{i_1 i_2}[0,1]$, i_1, $i_2 \in \{0,1,2\}$. Let $i_1 i_2 \dots i_k$ be the longest word such that $B_{2r}(x) \subseteq T_{i_1} T_{i_2} \dots T_{i_k}[0,1]$. Then $k \geq 2$, and $T_{i_1} T_{i_2} \dots T_{i_k}(q_j) \in B_{2r}(x)$ for some $j \in \{1,2\}$. Thus writing

$$\begin{aligned} B_{2r}(x) &= T_{i_1} T_{i_2} \dots T_{i_k} \widetilde{B_2} \\ B_r(x) &= T_{i_1} T_{i_2} \dots T_{i_k} \widetilde{B_1} \end{aligned}$$

we have q_1 or $q_2 \in \widetilde{B_2}$. Hence for any $i \in \{0,1,2\}$, $T_i \widetilde{B_2}$ contains a 'junction point.' It follows from the above that

$$\mu(T_i \widetilde{B_2}) \leq M\mu(T_i \widetilde{B_1})$$

holds for any $i = 0$, 1, 2. Hence by (4.1), we conclude that

$$\begin{aligned} \mu(B_{2r}(x)) &= \mu(T_{i_1} T_{i_2} \dots T_{i_k} \widetilde{B_2}) \\ &= (i_1\text{-th row of the matrix } Q_{i_2} Q_{i_3} \cdots Q_{i_k}) \cdot \begin{pmatrix} \mu(T_0 \widetilde{B_2}) \\ \mu(T_1 \widetilde{B_2}) \\ \mu(T_2 \widetilde{B_2}) \end{pmatrix} \\ &= x\mu(T_0 \widetilde{B_2}) + y\mu(T_1 \widetilde{B_2}) + z\mu(T_2 \widetilde{B_2}) \\ &\leq xM\mu(T_0 \widetilde{B_1}) + yM\mu(T_1 \widetilde{B_1}) + zM\mu(T_2 \widetilde{B_1}) \\ &= M\mu(B_r(x)), \end{aligned}$$

if (x,y,z) is the i_1-th row of the matrix $Q_{i_2} Q_{i_3} \cdots Q_{i_k}$. This proves (4.5), completing our proof of the sufficiency. $\square$

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