MATH1010A-H Assignment 6 Due date 7-12-2015

1. Find $F'(x)$ for the following functions.

   (a) $F(x) = \int_{\pi}^{x} \frac{\cos y}{y}\, dy$

   (b) $F(x) = \int_{x}^{1} \sqrt{1+t^2}\, dt$

   (c) $F(x) = \int_{0}^{x^3} e^{u^2}\, du$

   (d) $F(x) = \int_{x}^{2x} (\ln t)^2\, dt$

   (e) $F(x) = \int_{x^2}^{x^3} e^{\cos u}\, du$

   (f) $F(x) = \int_{-\sqrt{\ln x}}^{\sqrt{\ln x}} \frac{\sin t}{t}\, dt$

2. Evaluate the limits below.

   (a) $\lim_{h\to 0} \frac{1}{h} \int_{0}^{\sin(h)} \sin(\sqrt{t^2+t^4})dt$

   (b) $\lim_{h\to 0} \frac{1}{h\sin(h)} \int_{0}^{h^2} e^{t^2} dt$

   (c) $\lim_{h\to 0} \frac{1}{h} \int_{-h}^{h} \left|\sqrt[3]{\sin^5(t)}\right| dt$

   (d) $\lim_{h\to 0^+} \frac{1}{\ln(1+h)} \int_{2}^{3h+2} \sqrt{t^6+2t^4+3t^2+4}dt$

3. Define $f : (0,+\infty) \longrightarrow \mathbb{R}$ by $f(x) = \int_{x^{-1}}^{x} \cos(\sqrt{xt})dt$ for any $x \in (0,+\infty)$.

   (a) Show that $f(x) = \frac{1}{x}\int_{1}^{x^2} \cos(\sqrt{u})du$ for any $x \in (0,+\infty)$.

   (b) Find the value of $f'(1)$.

4. Evaluate the first derivative of the functions of x below.

   (a) $\int_{-2}^{x^3} x\sqrt{t^4+t+1}dt.$

   (b) $\int_{-x}^{x} |\cos(t)|^{\frac{7}{2}} dt$

   (c) $\int_{x}^{x^2} \sin\left(\frac{t}{x^2}\right) dt$

   (d) $\int_{20}^{x} \left(\int_{10}^{u} \frac{dt}{1+t^4+\sin^4(t)}\right) du$

5. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[0,1]$. Further suppose that $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for any $x \in [0,1]$. Show that $f(x) = 0$ for any $x \in [0,1]$.

6. Evaluate the definite/indefinite integrals below:

   (a) $\int_0^3 \frac{2(x-1)}{(x^2+3)(x+1)^2}dx$

   (b) $\int_0^{\frac{\pi}{2}} (\sin(x)+\cos(x))^2 dx$

   (c) $\int x\sin^2(x^2)dx$

   (d) $\int_0^{\frac{\pi}{2}} \cos^6(x)dx$

   (e) $\int \frac{2dx}{\cot(x/2)+\tan(x/2)}$

   (f) $\int_0^{\pi} |\sin(2x)-\sin(x)|dx$

   (g) $\int_0^{\pi} ||\sin(2x)|-\sin(x)|\, dx$

   (h) $\int \left(\frac{1}{x}+\frac{1}{x^2}\right)\ln(x)dx$

   (i) $\int_0^{\frac{\pi}{4}} \sec^4(x)dx$

   (j) $\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\sin(x)}{1+\sin(x)}}dx$

   (k) $\int_0^{\frac{\pi}{6}} \sin(x)\tan(x)dx$

   (l) $\int \frac{1+\cos(x)}{x+\sin(x)}dx$

   (m) $\int \frac{x^5 dx}{x^3-1}$. (Try not to 'break up' x^3-1.)

7. Show that $\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}dx = \int_0^{\pi} \frac{(\pi - x)\sin(x)}{1+\cos^2(x)}dx$. Hence, or otherwise, evaluate both definite integrals.

8. (a) Show that $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4(x)}{\sin^4(x)+\cos^4(x)}dx$.

   (b) Hence, or otherwise, compute $\int_0^{\pi} \frac{\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx$.

   (c) Show that $\int_0^{\pi} \frac{x\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx = \int_0^{\pi} \frac{(\pi - x)\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx$. Hence, or otherwise, evaluate both definite integrals.

9. Let n be a positive integer.

   (a) Show that $\cos(x) + \cos(3x) + \cos(5x) + \cdots + \cos((2n-1)x) = \frac{\sin(2nx)}{2\sin(x)}$ whenever $\sin(x) \neq 0$.

   (b) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin(2nx)}{\sin(x)}dx$

10. (a) Let f be a non-negative continuous function on $[a, b]$. Define

$$F(x) = \int_a^x f(t)\,dt$$

   for $x \in [a, b]$.
   Show that F is an increasing function on $[a, b]$.
   Hence deduce that if $\int_a^b f(t)\,dt = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

   (b) Let g be a continuous function on $[a, b]$. If $\int_a^b g(x)u(x)\,dx = 0$ for any continuous function u on $[a, b]$, show that $g(x) = 0$ for all $x \in [a, b]$.

   (c) Let h be a continuous function on $[a, b]$. Define

$$A = \frac{1}{b-a}\int_a^b h(t)\,dt.$$

   (i) If $v(x) = h(x) - A$ for all $x \in [a, b]$, show that $\int_a^b v(x)\,dx = 0$.

   (ii) If $\int_a^b h(x)w(x)\,dx = 0$ for any continuous function w on $[a, b]$ satisfying $\int_a^b w(x)\,dx = 0$, show that $h(x) = A$ for all $x \in [a, b]$.

11. Let n be a positive integer.

   (a) Show that $\frac{1}{1-t^2} = (1 + t^2 + \cdots + t^{2n-2}) + \frac{t^{2n}}{1-t^2}$ for $t^2 \neq 1$.

   (b) For $-1 < x < 1$, show that

   (i) $\int_0^x \frac{t}{1-t^2}\,dt = \ln \frac{1}{\sqrt{1-x^2}}$.

   (ii) $\int_0^x \frac{t^{2n+1}}{1-t^2}\,dt = \ln \frac{1}{\sqrt{1-x^2}} - \left(\frac{x^2}{2} + \frac{x^4}{4} + \cdots + \frac{x^{2n}}{2n}\right)$.

   (c) Show that $0 \leq \ln 3 - \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k \leq \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}$.

   Hence evaluate $\sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{8}{9}\right)^k$.