Week 10: Unitary / Orthogonal Matrices, Bilinear Forms (textbook § 6.5, 6.8)

Spectral Theorems Revisited

Recall: $Q \in M_{n\times n}(\mathbb{F})$ is unitary / orthogonal ($\mathbb{F} = \mathbb{C}$) ($\mathbb{F} = \mathbb{R}$) $\iff$ $\boxed{QQ^* = Q^*Q = I}$

These matrices arise naturally as change of coordinate matrices.

Lemma: Let β and γ be orthonormal bases of a finite dim'l inner product space $(V, \langle , \rangle)$. Then, the change of coordinate matrix $Q = [I]_\beta^\gamma$ is unitary / orthogonal. ($\mathbb{F} = \mathbb{C}$) ($\mathbb{F} = \mathbb{R}$)

"**Proof:**" We "explain" the proof by the example that

$$V = \mathbb{R}^n, \quad \gamma = \text{standard basis}$$

$$\beta = \{v_1, v_2, \dots, v_n\} \quad \text{O.N.B.}$$

In standard coordinates,

$$v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \quad \dots, \quad v_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$$

then the change of basis matrix is

$$Q = [I]_\beta^\gamma = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \boxed{\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{matrix}} \end{pmatrix}$$

$$\begin{matrix} \| & \| & & \| \\ v_1 & v_2 & & v_n \end{matrix}$$

Taking transpose,

$$Q^t = \begin{pmatrix} \boxed{a_{11} \quad a_{21} \quad \cdots \quad a_{n1}} \\ \boxed{a_{12} \quad a_{22} \quad \cdots \quad a_{n2}} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \boxed{a_{1n} \quad a_{2n} \quad \cdots \quad a_{nn}} \end{pmatrix} \begin{matrix} = v_1^t \\ = v_2^t \\ \\ = v_n^t \end{matrix}$$

Question: When is $Q^tQ = I$?

$$Q^tQ = \begin{pmatrix} \boxed{a_{11} \quad a_{21} \quad \cdots \quad a_{n1}} \\ \boxed{a_{12} \quad a_{22} \quad \cdots \quad a_{n2}} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \boxed{a_{1n} \quad a_{2n} \quad \cdots \quad a_{nn}} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_n\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \cdots & \langle v_2, v_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \cdots & \langle v_n, v_n\rangle \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix} = I$$

$\Updownarrow$

$\beta = \{v_1, v_2, \ldots, v_n\}$ is orthonormal !!

□

Corollary: Suppose β, γ are O.N.B. for $(V, \langle\cdot,\cdot\rangle)$, and $T: V \to V$.

$$[T]_\beta = Q^{-1}[T]_\gamma Q = Q^*[T]_\gamma Q$$

where Q is unitary ($\mathbb{F} = \mathbb{C}$) or orthogonal ($\mathbb{F} = \mathbb{R}$).

Def^n: Two matrices $A, B \in M_{n\times n}(\mathbb{F})$ are unitarily ($\mathbb{F} = \mathbb{C}$) / orthogonally ($\mathbb{F} = \mathbb{R}$) equivalent if there exists a unitary ($\mathbb{F} = \mathbb{C}$) / orthogonal ($\mathbb{F} = \mathbb{R}$) matrix Q

s.t.

$$\boxed{A = Q^*BQ}$$

Therefore, A, B are unitarily / orthogonally equivalent iff they represent the same linear operator T under different orthonormal bases!

As a result, we can restate the Spectral Theorems and Schur's Lemma in matrix form.

Spectral Theorems: (Matrix Form)

Let $A \in M_{n\times n}(\mathbb{F})$. Then, A is unitarily / orthogonally ($\mathbb{F}=\mathbb{C}$) / ($\mathbb{F}=\mathbb{R}$) equivalent to a diagonal matrix if and only if A is normal ($\mathbb{F}=\mathbb{C}$) or self-adjoint ($\mathbb{F}=\mathbb{R}$).

Schur's Lemma: (Matrix Form)

Let $A \in M_{n\times n}(\mathbb{F})$. If the characteristic polynomial of A splits over $\mathbb{F}$, then A is unitarily / orthogonally ($\mathbb{F}=\mathbb{C}$) / ($\mathbb{F}=\mathbb{R}$) equivalent to an upper triangular matrix.

Example: Let

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}) \qquad \boxed{\mathbb{F}=\mathbb{R}}$$

Find an orthogonal matrix P s.t. $P^t A P$ is a diagonal matrix.

Solution: First of all, $A^t = A$ (i.e. A is self adjoint), so such a P exists. Finding P is equivalent to finding an orthonormal eigenbasis for A.

Finding eigenvalues: $\det(A - \lambda I) = 0$.

$$0 = \det\begin{pmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{pmatrix} = (4-\lambda)\det\begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} - 2\cdot\det\begin{pmatrix} 2 & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$+ 2\cdot\det\begin{pmatrix} 2 & 4-\lambda \\ 2 & 2 \end{pmatrix}$$

$$= (4-\lambda)\left[(4-\lambda)^2 - 4\right] - 2\left[2(4-\lambda) - 4\right]$$

$$+ 2\left[4 - 2(4-\lambda)\right]$$

$$= (4-\lambda)(\lambda^2 - 8\lambda + 12) - 8(2-\lambda)$$

$$\overset{(?)}{=} -(\lambda-4)(\lambda-6)(\lambda-2) + 8(\lambda-2)$$

$$= (\lambda-2)\left[-(\lambda^2 - 10\lambda + 24) + 8\right]$$

$$= (\lambda-2)\left[-(\lambda^2 - 10\lambda + 16)\right]$$

$$= -(\lambda-2)^2(\lambda-8).$$

The eigenvalues and multiplicities are

$$\boxed{\begin{aligned} \lambda_1 &= 2, & m_1 &= 2 \\ \lambda_2 &= 8, & m_2 &= 1 \end{aligned}}$$

Finding orthonormal eigenvectors:

$\boxed{\lambda_1 = 2}$ $$E_{\lambda_1} = N(A - 2I) = N\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \operatorname{span}\Bigg\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{\text{NOT orthonormal !!}} \Bigg\}$$

extra step

Apply Gram-Schmidt!

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

Hence, $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is an O.N.B. for E_{λ_1}.

$\boxed{\lambda_2 = 8}$ $E_{\lambda_2} = N(A - 8I) = N\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$

Hence, $\left\{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is an O.N.B. for E_{λ_2}.

Putting all these together,

$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is an orthonormal eigenbasis for $\mathbb{R}^3$.

The required orthogonal matrix P is thus given by

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

and $$P^tAP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Midterm 2 up to here ——— □

Bilinear Forms

Recall that an inner product was a 2-variable scalar-valued function

$$\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{F}$$

which satisfies some properties (Ex: What are they?).

We know that $\langle \, , \, \rangle$ is linear in the 1st variable but only conjugate linear in the 2nd variable! (Note: There is no such different though when $\mathbb{F} = \mathbb{R}$.)

Now, we study 2-variable functions when are more "symmetric" in the two variables.

Def$^{\underline{n}}$: Let V be a vector space over $\mathbb{F}$.

A bilinear form is a function

$$H : V \times V \longrightarrow \mathbb{F}$$

which is linear in each variable:

$$\boxed{\begin{aligned} H(a_1x_1+a_2x_2, y) &= a_1 H(x_1,y) + a_2 H(x_2,y) \\ H(x, a_1y_1+a_2y_2) &= a_1 H(x,y_1) + a_2 H(x,y_2) \end{aligned}}$$

for $\underline{\underline{\text{all}}}$ $a_1, a_2 \in \mathbb{F}$, $x_1, x_2, x, y_1, y_2, y \in V$.

Remark: Even though many interesting applications are concerned with general fields $\mathbb{F}$ (e.g. $\mathbb{Z}_2$), for simplicity, we will mainly be focusing on the case $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

Example: If $\mathbb{F} = \mathbb{R}$, any inner product $\langle\,,\rangle$ is a bilinear form.

Example: Given any $A \in M_{n\times n}(\mathbb{R})$, the function

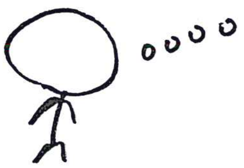
$$H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

defined by $\boxed{H(x,y) = x^tAy}$ is a bilinear form.

e.g. $$H\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = (x_1 \;\; x_2)\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 2x_1y_1 + 3x_1y_2 + 4x_2y_1 - x_2y_2$$

Given any matrix $A \in M_{n\times n}(\mathbb{F})$, I can associate to it a bilinear form H as above. Does all bilinear forms come from some matrix?



The Space of Bilinear Forms $\mathcal{B}(V)$

Question: What can we do with bilinear forms?

- We can add two bilinear forms:

$$(H_1 + H_2)(x, y) := H_1(x,y) + H_2(x,y)$$

- We can scalar-multiply:

$$(\lambda H)(x, y) := \lambda H(x,y)$$

FACT: The space of bilinear forms $\mathcal{B}(V)$ on a given vector space V forms a vector space (over the same field $\mathbb{F}$ as V).

Question: What is $\dim \mathcal{B}(V)$?

Answer: $\dim \mathcal{B}(V) = n^2$ if $\dim V = n$.

In fact, there exists a vector space isomorphism

$$\mathcal{B}(V) \xrightarrow[\text{not canonical}]{\cong} M_{n\times n}(\mathbb{F}) \qquad \text{if } \dim_{\mathbb{F}} V = n$$

! Does it look familiar? !

$$\mathcal{L}(V) := \{T : V \to V \text{ linear}\} \xrightarrow[\text{not canonical}]{\cong} M_{n\times n}(\mathbb{F})$$

We have such a correspondence once we choose a basis β

(hence not entirely "natural")

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis for V.

Using bilinearity,

$$H(\underbrace{\sum_{i=1}^n a_i v_i}_{= x}, \underbrace{\sum_{j=1}^n b_j v_j}_{= y}) = \sum_{i,j=1}^n a_i b_j \underbrace{H(v_i, v_j)}_{\text{these } n\times n = n^2 \text{ "coefficients" determine } H \text{ completely.}}$$

Def$^{\underline{n}}$: Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis for V. If $H \in \mathcal{B}(V)$,

$$\psi_\beta(H) := \begin{pmatrix} H(v_1,v_1) & H(v_1,v_2) & \cdots & H(v_1,v_n) \\ H(v_2,v_1) & H(v_2,v_2) & \cdots & H(v_2,v_n) \\ \vdots & \vdots & \ddots & \vdots \\ H(v_n,v_1) & H(v_n,v_2) & \cdots & H(v_n,v_n) \end{pmatrix} = \big(H(v_i,v_j) \big)$$

is the matrix representation of H w.r.t. β.

Example: Consider $H : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ to be the "determinant":

$$H\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) := \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

By properties of det., H is a bilinear form.

Fix the standard basis $\beta = \{e_1, e_2\}$

$$H(e_1,e_1) = \det\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

$$H(e_1,e_2) = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$H(e_2,e_1) = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

$$H(e_2,e_2) = \det\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0$$

$$\boxed{\psi_\beta(H) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

skew-symmetric
"$A^t = -A$"

By a very similar argument in establishing $\mathcal{L}(V) \cong M_{n\times n}(\mathbb{F})$, we have:

Theorem: The map $\psi_\beta : \mathcal{B}(V) \xrightarrow{\cong} M_{n\times n}(\mathbb{F})$ is an isomorphism between vector spaces over $\mathbb{F}$.

Proof: Exercise (see textbook Thm. 6.32 if you got stuck.)

Change of Basis Formula

Recall that for a linear operator $T : V \to V$, if we have two bases β and γ for V, then the matrices $[T]_\beta$, $[T]_\gamma$ are related by

$$(*)_T \quad \boxed{[T]_\gamma = Q^{-1} [T]_\beta Q}$$

where Q is an invertible matrix.

? What about for bilinear forms ?

<u>Theorem</u>: Given a bilinear form $H \in \mathcal{B}(V)$, if β and γ are two bases for V, then the matrices $\psi_\beta(H)$ and $\psi_\gamma(H)$ are related by

$$(*)_H \quad \boxed{\psi_\gamma(H) = Q^t \psi_\beta(H) Q}$$

where $Q = [I]_\gamma^\beta$ is the invertible change of basis matrix from γ to β.

<u>Caution</u>! The matrix Q in $(*)_H$ may not be orthogonal, i.e. $Q^t \neq Q$ in general.

<u>Note</u>: Comparing $(*_T)$ and $(*_H)$, we see that although both linear operators T and bilinear forms H are represented by matrices, they transform differently when we change basis. Hence T and H are somewhat different in some fundamental way. In the language of "tensors",

$T \in V \otimes V^*$ but $H \in V^* \otimes V^*$.

Proof of $(*_H)$:

Let Q be the change of coordinate matrix from γ to β,

i.e. $(\#)$ $\boxed{[v]_\beta = Q[v]_\gamma}$ for all $v \in V$.

Now, by definition of matrix representation of H, we have

$$H(x,y) = [x]_\beta^t \,\psi_\beta(H)\,[y]_\beta \quad \& \quad H(x,y) = [x]_\gamma^t \boxed{\psi_\gamma(H)}\,[y]_\gamma$$

$$\overset{(\#)}{=} [x]_\gamma^t \boxed{Q^t \psi_\beta(H) Q}\,[y]_\gamma$$

compare these expressions!

$\square$

Because of $(*_H)$, we identify matrices which represent the same bilinear form H but in different bases.

Def$^{\text{n}}$: Two matrices $A, B \in M_{n\times n}(\mathbb{F})$ are congruent if there exists invertible $Q \in M_{n\times n}(\mathbb{F})$ s.t. $\boxed{B = Q^t A Q}$

Diagonalizability

Since we can represent bilinear forms H as matrices, we have a similar notion of "diagonalizability":

Def$^{\text{n}}$: $H \in \mathcal{B}(V)$ is diagonalizable iff there exists a basis β for V s.t. $\psi_\beta(H)$ is diagonal.

In terms of matrices, it is the same as asking:

> Given $A \in M_{n\times n}(\mathbb{F})$, does there exist an invertible $Q \in M_{n\times n}(\mathbb{F})$ s.t. $Q^t A Q$ is diagonal?

! This is different from our usual notion of diagonalizing a matrix!

Symmetric Bilinear Forms

Fixing a basis β for V, we have a 1-1 correspondence

$$\begin{array}{ccc} \mathcal{B}(V) & \xleftrightarrow{\cong} & M_{n\times n}(\mathbb{F}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ H & \longleftrightarrow & \psi_\beta(H) \end{array} \qquad \dim V = n$$

As symmetric matrices are particularly nice in many aspects, we ask: $\boxed{\text{For which } H \text{ is } \psi_\beta(H) \text{ a symmetric matrix?}}$

Def$^{\underline{n}}$: $H \in \mathcal{B}(V)$ is symmetric iff $\boxed{H(x,y) = H(y,x) \quad \forall x, y \in V}$

FACT: $H \in \mathcal{B}(V)$ symmetric $\iff$ $\psi_\beta(H)$ symmetric

Pf: Exercise!

The following is the most important result about symmetric bilinear forms:

Theorem: Let V be a finite dim'l vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$$\boxed{H \in \mathcal{B}(V) \text{ is diagonalizable}} \iff \boxed{H \text{ is a symmetric bilinear form}}$$

Caution: This is different from the Spectral Theorems we discussed since there is NO inner product $\langle \cdot , \cdot \rangle$ involved!

Proof: By induction on $n = \dim V$.

$n = 1$: trivial

Assume theorem is true for $n = k-1$.

Suppose now $n = \dim V = k$. WLOG, assume $H \neq 0$.

Claim: $H(x,x) \neq 0$ for some $x \in V$

Proof of claim: $H \neq 0 \Rightarrow H(u,v) \neq 0$ for some $u, v \in V$

Let $x = u + v$. Suppose $H(u,u) = H(v,v) = 0$. Otherwise we are done. Then

$$H(x,x) = H(u+v, u+v)$$

H symmetric

(bilinearity) $$= H(u,u) + H(u,v) + H(v,u) + H(v,v)$$

$$= 0 + 2H(u,v) + 0 \neq 0.$$

□

Fix some $0 \neq x \in V$ s.t. $H(x,x) \neq 0$.

Consider the linear map $L_x : V \to \mathbb{F}$ defined by "fixing one of the variable to be x", i.e.

$$\boxed{L_x(y) := H(x,y)}$$

Since $L_x(x) = H(x,x) \neq 0$ by our choice of x, L_x is onto.

By rank-nullity theorem,

$$\dim N(L_x) = \dim V - \dim R(L_x) = k-1.$$

Therefore, $H : \underbrace{N(L_x)}_{\dim = k-1} \times N(L_x) \to \mathbb{F}$ is again a symmetric bilinear form

and is thus diagonalizable by some basis $\{v_1, \dots, v_{k-1}\}$ for $N(L_x)$.

Then, $\{v_1, \dots, v_{k-1}, x\}$ is a basis for V which diagonalize H.

□

Quadratic Forms

Recall that an inner product induces a norm :

$$\langle \, , \, \rangle \rightsquigarrow \| x \|^2 := \langle x , x \rangle$$

(Note: But NOT vice versa, unless we have a "nice" norm (see Exercise #27 in 6.1))

We have something similar for bilinear forms which are symmetric :

Def$^{\underline{n}}$: For a symmetric bilinear form $H \in \mathcal{B}(V)$ we associate a quadratic form $K : V \to \mathbb{F}$ where

$$\boxed{K(x) := H(x, x)} \quad \text{for all } x \in V$$

Example : (Conic sections)

Remember that any symmetric bilinear form H on $\mathbb{R}^2$ is represented by a 2×2 symmetric real matrix :

$$H\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = (x_1 \;\; x_2) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The quadratic form associated to H is simply

$$K\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = (x \;\; y) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A x^2 + 2 B xy + C y^2$$

which is just a homogeneous quadratic polynomial in x and y.

For example, if $A = 1, B = 2, C = 5$.

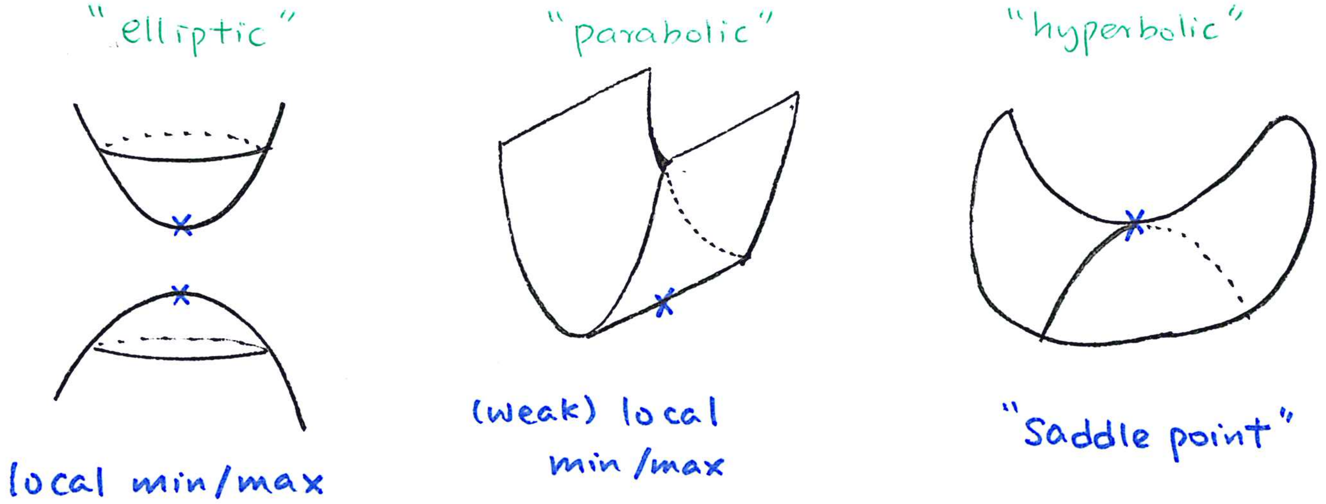
$$K(x, y) = x^2 + 4xy + 5y^2 = (x + 2y)^2 + y^2 = u^2 + v^2$$

if we change coordinates $\begin{cases} u = x + 2y \\ v = y \end{cases}$

By a linear change of coordinates, we can always express it in one of the following forms (Exercise: Can you prove this?)

$$K(u,v) = \begin{cases} u^2+v^2 \text{ or } -u^2-v^2 & \text{"elliptic"} \\ \pm u^2 & \text{"parabolic"} \\ -u^2+v^2 & \text{"hyperbolic"} \end{cases}$$

Locally near $(0,0)$, their graphs look like



Example: Does the function $f(x,y) = 5x^2 + 4xy + 2y^2$ have a local min./max or saddle point at $(0,0)$?

Solution: By "completing the square",

$$f(x,y) = 5x^2 + 4xy + 2y^2 = 3x^2 + 2(x+y)^2$$

$$= u^2 + v^2 \qquad \text{if we let } u = \sqrt{3}x, \ v = \sqrt{2}(x+y)$$

Hence, $(0,0)$ is a local minimum.

Alternatively, f is represented by the symmetric 2×2 matrix

$$\begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \end{array}$$ which has eigenvalues 1 and 6 $\xRightarrow[\text{positive}]{\text{both}}$ local min

Char. eq$^{\underline{n}}$: $(5-\lambda)(2-\lambda) - 4 = 0 \Rightarrow \lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 1$ or 6

$\underline{\qquad\qquad}$ $\square$