MATH 2221 B Mathematics Laboratory II

Midterm

Name: _____   Student ID.: _____

In this test, the color version of this assignment can be obtained from the **H:** drive.

Please read the following carefully:

General Guidelines for Test Submission.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your test). Please write your answers using a black or blue pen, NOT any other color or a pencil.

- Please go to the class and lab indicated by your registered course code via the CUSIS system. If you go to a different lab than the one you are registered for, you will not receive credit for the test even if you completed it.

- The use of cellular phones/graphing calculators/iPads will NOT be permitted during the test. Please do not use our lab computer to recharge your cellar phone battery. No photo taking is allowed in the test.

- If you are found cheating in the tests, you will automatically get an F grade in this test and your act will be reported to the Department for necessary disciplinary actions.

- During the test, you can freely access the `help` command. All internet links for the outside world will be disconnected, including our course website. *You are allowed to bring book, lecture notes, formula sheets, etc. to the test. But they should be put on the desk before the test begins. Once the test starts, you are not allowed to put any additional things on your desk or take anything out of your bag.*

- Answer all questions. Show work to justify all answers.

- The quiz lasts 105 minutes.

- Write your solutions on a double-sided printout of this pdf file. Try to fit your answers inside the available space.

- Please save all your codes in the `MATLAB` m-file. We will take not any responsibility for your m-files if you did not save your works. There will be no way to make up m files after the test.

- If you finish early, please remain in your seat so that you do not disturb others.

- When time is called, stop writing and pass your exam to the aisle. Please stay in your seat until all of the exams are collected.

- Please sign and date the statement of Academic Honesty.

Please read and sign the following declaration before handing in your test. Otherwise, no marks will be given.

1 (15 marks)		4 (15 marks)	
2 (15 marks)		5 (20 marks)	
3 (15 marks)		6 (20 marks)	
		Total	/100 pts

1. (15 marks) Open `q1.m`, follow the instructions and fill in the blanks.

   Any given matrix $\mathbf{A}$ can be factorized into a product of two matrices $\mathbf{L}$ and $\mathbf{U}$, where $\mathbf{L}$ is a lower triangular matrix, and $\mathbf{U}$ is an upper triangular matrix.

   (a) Let
   $$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 4 & -2 \\ -1 & -1 & \frac{1}{2} \end{pmatrix}$$
   Create $\mathbf{A}$ and use MATLAB built-in function `lu` to compute $\mathbf{L}$ and $\mathbf{U}$.

   (b) Consider the following system of equations:
   $$\begin{cases} 3x - y + 2z = 1 \\ 4x + 8y - 4z = -4 \\ -x - y + \frac{1}{2}z = 0 \end{cases}$$

   i. The above system can be formulated as $\mathbf{A}\mathbf{x} = \mathbf{b}$, create $\mathbf{b}$.
   ii. Solve for $\mathbf{y}$ in the equation $\mathbf{L}\mathbf{y} = \mathbf{b}$.
   iii. Solve for $\mathbf{x}$ in the equation $\mathbf{U}\mathbf{x} = \mathbf{y}$

   Remark: $\mathbf{x}$ is the solution for the system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

```
clear all;

A = [3,-1,2;2,4,-2;-1,-1,0.5];
[L,U] = lu(A);

b = [1;-2;0];

y = L\b;

x = U\y;
```

2. (15 marks) Use **subplot** and **polar** to plot the curves $r = \theta^k$.

   Instructions:

   (a) Define $(x, y) = (\theta^k \cos\theta, \theta^k \sin\theta)$, with $\theta \in [0, 2\pi]$ increment $0.02 \times \pi$, for $k = 3, 5, 7, 9$;

   (b) Plot all the graphs in (a) using red solid line. All the graphs should be put in the **SAME** figure. For $k = 3, k = 5$, these two graphs are shown in the first row of figure, while the graphs for $k = 7, k = 9$ are beneath them;

   (c) Each of the graphs should contain a title, i.e., "$r = \theta^3$", "$r = \theta^5$",..., respectively.

Hint: You may use "\theta^k" for expressing θ^k in the title.

Write a script named `q2.m` and save it in the directory `H.` and save your graph by using the command `print('q2.jpg','-depsc')`.

Solution of `q2.m`

```
% q2

clear all
x = [0:0.02:2]*pi;
subplot(2,2,1); polar(x,x.^3,'red');
title('r=\theta^3');
subplot(2,2,2); polar(x,x.^5,'red');
title('r=\theta^5');
subplot(2,2,3); polar(x,x.^7,'red');
title('r=\theta^7');
subplot(2,2,4); polar(x,x.^9,'red');
title('r=\theta^9');
print('q2.jpg','-depsc')
>>
```

3. (15 marks) The famous **Mobius strip** can be represented by a 2-dimensional parametrization:

$$x(u,v) = (1 + \frac{v}{2}\cos\frac{u}{2})\cos u$$
$$y(u,v) = (1 + \frac{v}{2}\cos\frac{u}{2})\sin u$$
$$z(u,v) = \frac{v}{2}\sin\frac{u}{2}$$

where $u \in [0, 2\pi]$, $v \in [-1, 1]$. Write a script named `q3.m` to do the followings:

(a) Use `meshgrid` to generate the grid of **u** from 0 to 2π, with step size $\frac{\pi}{50}$, **v** from -1 to 1, with step size 0.1.

(b) Compute x, y and z.

(c) Use `mesh` to visualize the Mobius strip. Set view from azimuth -15 and elevation 61 and show the color bar. Save your graph by using the command `print('q3c.eps','-depsc')`. Your figure should be exactly the same as Figure 1.

(d) Use `contour3` to plot the surface with 30 contour lines. Set view from azimuth -15 and elevation 61. Save your graph by using the command `print('q3d.eps','-depsc')`. Your figure should be exactly the same as Figure 2.

```
clear all;
close all;

[u,v] = meshgrid(-1:0.2:1,0:pi/10:2*pi);

x = cos(v).*(1+u/2.*cos(v/2));
y = sin(v).*(1+u/2.*cos(v/2));
z = u/2.*sin(v/2);

figure;mesh(x,y,z);colorbar;view(-15,61);print('q3c.eps','-depsc')
figure;contour3(x,y,z,25);view(-15,61);print('q3d.eps','-depsc');
```
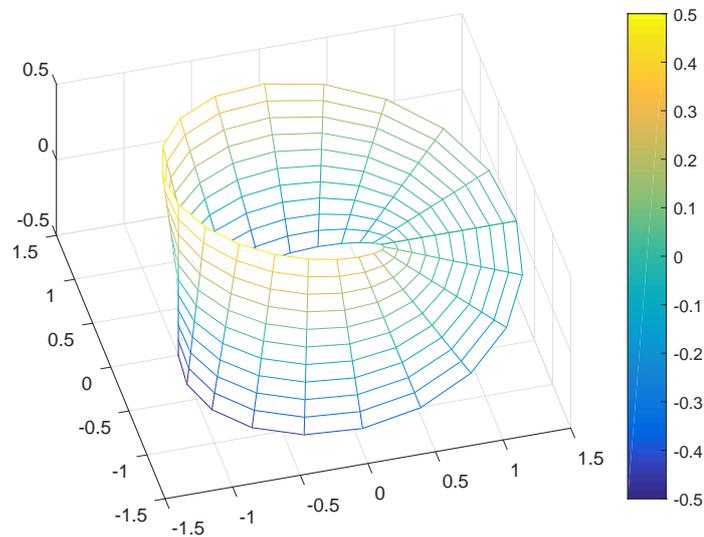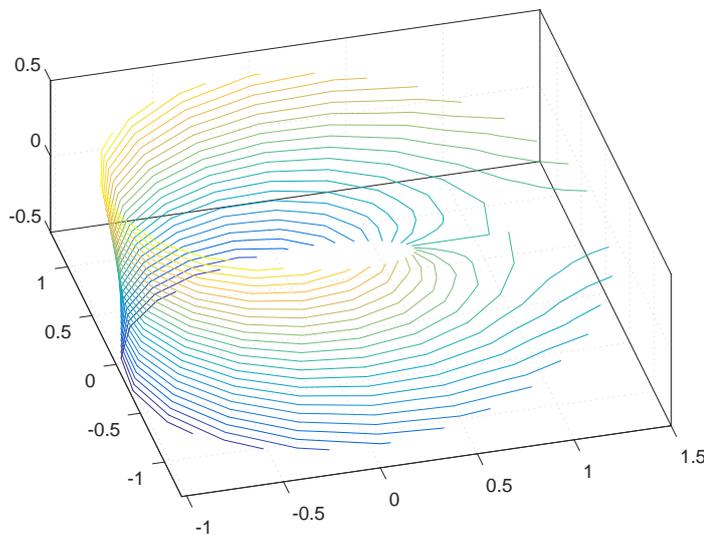

Figure 1: q3c.eps

Figure 2: q3d.eps

4. (15 marks) In MATLAB, the eigenvectors and eigenvalues of a matrix A can be found by the command [U,D] = eig(A), where each column vector of U is an eigenvector, each diagonal entry of D is the corresponding eigenvalue and U, D satisfy

$$A = UDU'.$$

Let A be a 256-by-256 Hadamard matrix, which can be created by command A=hadamard(256);

Do the followings and save your script and summary in a single m-file named q4abc.m. The summary should be in the form of comments, like

```
% So the matrix U is not an orthogonal matrix.
```

(a) (6 marks) Create the matrix A and find U and D (do not need to print them).

```
A = hadamard(256);
[U,D] = eig(A);
```

(b) (6 marks) Use basic matrix operations, transpose, MATLAB built-in functions norm and diag only to check (i) whether U is an orthogonal matrix, and (ii) whether D is a diagonal matrix.
Hint: A matrix B is a zero matrix if and only if norm(B) is zero.

```
>> norm( U'*U - eye(256) )

ans =

    4.4828e-15


    % so U is an orthogonal matrix


>> norm( D - diag(diag(D)) )

ans =

     0

    % so D is an diagonal matrix
```

(c) (3 marks) Check whether the columns of U are eigenvectors of A.
Hint: A matrix B is a zero matrix if and only if norm(B) is zero.

```
>> norm( A*U - U*D )

ans =

   6.9561e-14

   %% so the columns of U are eigenvectors of A
```

5. (20 Marks) Define a sequence $\{x_n\}$ by

$$x_n = \frac{x_{n-1}x_{n-2} + 5}{x_{n-1} + x_{n-2}}$$

where $x_1 = 8, x_2 = 18$.

(a) (10 Marks) Use a for-loop to compute x_n for $n = 1, 2, 3, 4, 5, 6$. You should store the values in a vector x. Save your script in an m-file named q5a.m.

```
>>  x=zeros(6,1);
    x(1)=8;
    x(2) = 18;
    for i=3:6
        x(i)= (x(i-1)*x(i-2) + 5 ) / (x(i-1) + x(i-2));
    end
    x


x =

    8.0000
   18.0000
    5.7308
    4.5575
    3.0246
    2.4775
```

(b) (10 Marks) Guess the limit of x_n as $n \to \infty$. Store the number you guess in a variable guess. You may assume that $\lim_{n\to\infty} x_n$ exists. Use while-loop to compute the smallest n such that the distance between x_n and the number you guess is smaller than 10^{-13}. Save your script in an m-file named q5b.m.

```
>>  guess = sqrt(5);
    n = 2;
    x(1)=8;
    x(2) = 18;
    while abs(x(n)-guess)>=1e-13
        n = n + 1;
        x(n)= (x(n-1)*x(n-2) + 5 ) / (x(n-1) + x(n-2));
    end
    n


n =

    11
```

6. (20 marks) Let
$$C_n(x) = \sum_{k=1}^{n}(-1)^{k+1}\frac{x^k}{k}, \quad \text{for all } x.$$

(a) Write a script named q6.m to compute $C_n(x)$ for any given natural number n and real $x \in (-1, 1)$.

Instructions:

   i. Input n by showing "Input n: ";
   ii. Input x by showing "Input x: ";
   iii. Compute and display $C_4, C_8, C_{12}, \cdots, C_{4j}$ (where $4j \le n$) like the sample output;
   iv. Finally, show the value of C_n.

Save your script q6.m in the directory H: .

Sample Output

```
>> q6
Input n: 17
Input x: 1/3
Step 4, C_4 =  0.28704;
Step 8, C_8 =  0.28768;
Step 12, C_12 =  0.28768;
Step 16, C_16 =  0.28768;

C_n =

    0.2877
```

Solution of lab_7_q2.m

```
% q6
clear

n = input('Input n: ');
x = input('Input x: ');
C_n = 0;
for k = 1 : n
    C_n = C_n +  (-1)^(k+1)*x^(k) / k;
    if fix(k/4) == k/4
        disp(['Step ',num2str(k),', C_',num2str(k), ' =  ', ...
            num2str(C_n), ';   '])
    end
end
C_n
```

(b) i. Compute $C_{100}(1/2)$ and compare with $\log(3/2)$. Are these two values very close?

ii. Try some other x with large enough n, do you think $C_n(x)$ converges for $|x| < 1$ as $n \to \infty$? If so, what should be the limit function? (just write down the conclusion)

```
i) 0.4055, Yes, two values are very close and the
    absolute error is 1.1102e-16;
ii) Yes. the limit function is log(x+1).
```