maximum value of $|f(z)|$ *on* C_R, *then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \qquad (n = 1, 2, \ldots). \tag{2}$$

Inequality (2) is called ***Cauchy's inequality*** and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \qquad (n = 1, 2, \ldots),$$

in the theorem in Sec. 55 when n is a positive integer. We need only apply the theorem in Sec. 47, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} 2\pi R \qquad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; (b) $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\,dz$; (c) $\displaystyle\int_C \frac{z\,dz}{2z + 1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; (e) $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; (b) $\pi i/4$; (c) $-\pi i/2$; (d) 0; (e) $i\pi \sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2 + 4}$; (b) $g(z) = \dfrac{1}{(z^2 + 4)^2}$.

Ans. (a) $\pi/2$; (b) $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\,ds \qquad (|z| \ne 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\,ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following the procedure used in Sec. 56, prove that the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}\ (-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

8. Show that $P_n(-1) = (-1)^n\ (n = 0, 1, 2, \ldots)$, where $P_n(z)$ are the Legendre polynomials in Example 3, Sec. 55.

Suggestion: Note that

$$\frac{(s^2 - 1)^n}{(s + 1)^{n+1}} = \frac{(s - 1)^n}{s + 1}.$$

9. Follow the steps below to verify the expression

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s - z)^3}$$

in Sec. 56.

(a) Use expression (2) in Sec. 56 for $f'(z)$ to show that

$$\frac{f'(z + \Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s - z)^3} = \frac{1}{2\pi i}\int_C \frac{3(s - z)\Delta z - 2(\Delta z)^2}{(s - z - \Delta z)^2 (s - z)^3} f(s)\,ds.$$

(b) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 56 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part (a) is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d - |\Delta z|)^2 d^3} L.$$

(c) Use the results in parts (a) and (b) to obtain the desired expression for $f''(z)$.

10. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 57) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

58. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality in Theorem 3 of Sec. 57 can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here, which is known as ***Liouville's theorem***, states this result in a slightly different way.

Theorem 1. *If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

To start the proof, we assume that f is as stated and note that since f is entire, Theorem 3 in Sec. 57 can be applied with any choice of z_0 and R. In particular, Cauchy's inequality (2) in that theorem tells us that when $n = 1$,

$$|f'(z_0)| \leq \frac{M_R}{R}. \tag{1}$$

Moreover, the boundedness condition on f tells us that a nonnegative constant M exists such that $|f(z)| \leq M$ for all z; and, because the constant M_R in inequality (1) is always less than or equal to M, it follows that

$$|f'(z_0)| \leq \frac{M}{R}, \tag{2}$$

where R can be arbitrarily large. Now the number M in inequality (2) is independent of the value of R that is taken. Hence that inequality holds for arbitrarily large values of R only if $f'(z_0) = 0$. Since the choice of z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function, according to the theorem in Sec. 25.

The following theorem is called the ***fundamental theorem of algebra*** and follows readily from Liouville's theorem.

Theorem 2. *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n ($n \geq 1$) has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

The proof here is by contradiction. Suppose that $P(z)$ is *not* zero for any value of z. Then the quotient $1/P(z)$ is clearly entire. It is also bounded in the complex plane.

To see that it is bounded, we first recall statement (6) in Sec. 5. Namely, there is a positive number R such that

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n} \quad \text{whenever } |z| > R.$$

So $1/P(z)$ is bounded in the region *exterior to* the disk $|z| \leq R$. But $1/P(z)$ is continuous on that closed disk, and this means that $1/P(z)$ is bounded there too (Sec. 18). Hence $1/P(z)$ is bounded in the entire plane.

It now follows from Liouville's theorem that $1/P(z)$, and consequently $P(z)$, is constant. But $P(z)$ is not constant, and we have reached a contradiction.*

The fundamental theorem tells us that any polynomial $P(z)$ of degree n ($n \geq 1$) can be expressed as a product of linear factors:

$$P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n), \tag{3}$$

where c and z_k ($k = 1, 2, \ldots, n$) are complex constants. More precisely, the theorem ensures that $P(z)$ has a zero z_1. Then, according to Exercise 8, Sec. 59,

$$P(z) = (z - z_1)Q_1(z),$$

where $Q_1(z)$ is a polynomial of degree $n - 1$. The same argument, applied to $Q_1(z)$, reveals that there is a number z_2 such that

$$P(z) = (z - z_1)(z - z_2)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n - 2$. Continuing in this way, we arrive at expression (3). Some of the constants z_k in expression (3) may, of course, appear more than once, but it is clear that $P(z)$ can have no more than n *distinct* zeros.

59. MAXIMUM MODULUS PRINCIPLE

In this section, we derive an important result involving maximum values of the moduli of analytic functions. We begin with a needed lemma.

Lemma. *Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.*

To prove this, we assume that f satisfies the stated conditions and let z_1 be any point other than z_0 in the given neighborhood. We then let ρ be the distance between z_1 and z_0. If C_ρ denotes the positively oriented circle $|z - z_0| = \rho$, centered at z_0 and

*For an interesting proof of the fundamental theorem of algebra using the Cauchy–Goursat theorem, see R. P. Boas, Jr., *Amer. Math. Monthly*, Vol. 71, No. 2, p. 180, 1964.

where each N_k has center z_k and radius d, we see that f is analytic in each of these neighborhoods, which are all contained in D, and that the center of each neighborhood N_k $(k = 1, 2, \ldots, n)$ lies in the neighborhood N_{k-1}.

Since $|f(z)|$ was assumed to have a maximum value in D at z_0, it also has a maximum value in N_0 at that point. Hence, according to the preceding lemma, $f(z)$ has the constant value $f(z_0)$ throughout N_0. In particular, $f(z_1) = f(z_0)$. This means that $|f(z)| \le |f(z_1)|$ for each point z in N_1; and the lemma can be applied again, this time telling us that

$$f(z) = f(z_1) = f(z_0)$$

when z is in N_1. Since z_2 is in N_1, then, $f(z_2) = f(z_0)$. Hence $|f(z)| \le |f(z_2)|$ when z is in N_2; and the lemma is once again applicable, showing that

$$f(z) = f(z_2) = f(z_0)$$

when z is in N_2. Continuing in this manner, we eventually reach the neighborhood N_n and arrive at the fact that $f(z_n) = f(z_0)$.

Recalling that z_n coincides with the point P, which is any point other than z_0 in D, we may conclude that $f(z) = f(z_0)$ for *every* point z in D. Inasmuch as $f(z)$ has now been shown to be constant throughout D, the theorem is proved.

If a function f that is analytic at each point in the interior of a closed bounded region R is also continuous throughout R, then the modulus $|f(z)|$ has a maximum value somewhere in R (Sec. 18). That is, there exists a nonnegative constant M such that $|f(z)| \le M$ for all points z in R, and equality holds for at least one such point. If f is a constant function, then $|f(z)| = M$ for all z in R. If, however, $f(z)$ is not constant, then, according to the theorem just proved, $|f(z)| \ne M$ for any point z in the interior of R. We thus arrive at an important corollary.

Corollary. *Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.*

When the function f in the corollary is written $f(z) = u(x, y) + iv(x, y)$, *the component function $u(x, y)$ also has a maximum value in R which is assumed on the boundary of R and never in the interior*, where it is harmonic (Sec. 27). This is because the composite function $g(z) = \exp[f(z)]$ is continuous in R and analytic and not constant in the interior. Hence its modulus $|g(z)| = \exp[u(x, y)]$, which is continuous in R, must assume its maximum value in R on the boundary. In view of the increasing nature of the exponential function, it follows that the maximum value of $u(x, y)$ also occurs on the boundary.

Properties of *minimum* values of $|f(z)|$ and $u(x, y)$ are similar and treated in the exercises.

EXAMPLE. Consider the function $f(z) = (z + 1)^2$ defined on the closed triangular region R with vertices at the points

$$z = 0, \quad z = 2, \quad \text{and} \quad z = i.$$

A simple geometric argument can be used to locate points in R at which the modulus $|f(z)|$ has its maximum and minimum values. The argument is based on the interpretation of $|f(z)|$ as the square of the distance d between -1 and any point z in R:

$$d^2 = |f(z)| = |z - (-1)|^2.$$

As one can see in Fig. 74, the maximum and minimum values of d, and therefore $|f(z)|$, occur at boundary points, namely $z = 2$ and $z = 0$, respectively.

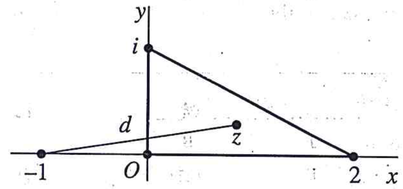

FIGURE 74

EXERCISES

1. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound u_0; that is, $u(x, y) \le u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.
   Suggestion: Apply Liouville's theorem (Sec. 58) to the function $g(z) = \exp[f(z)]$.

2. Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \ne 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value m* in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 59) to the function $g(z) = 1/f(z)$.

3. Use the function $f(z) = z$ to show that in Exercise 2 the condition $f(z) \ne 0$ anywhere in R is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ *can* reach its minimum value at an interior point when the minimum value is zero.

4. Let R region $0 \le x \le \pi, 0 \le y \le 1$ (Fig. 75). Show that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = (\pi/2) + i$.
   Suggestion: Write $|f(z)|^2 = \sin^2 x + \sinh^2 y$ (see Sec. 37) and locate points in R at which $\sin^2 x$ and $\sinh^2 y$ are the largest.

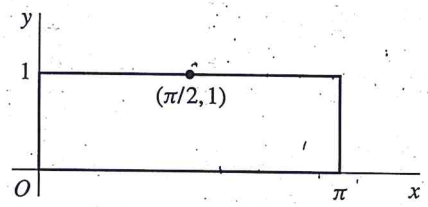

FIGURE 75

5. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x, y)$ has a minimum value in R which occurs on the boundary of R and never in the interior. (See Exercise 2.)

6. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \le x \le 1, 0 \le y \le \pi$. Illustrate results in Sec. 59 and Exercise 5 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.

Ans. $z = 1, z = 1 + \pi i$.

7. Let the function $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed bounded region R, and suppose that it is analytic and not constant in the interior of R. Show that the component function $v(x, y)$ has maximum and minimum values in R which are reached on the boundary of R and never in the interior, where it is harmonic.

Suggestion: Apply results in Sec. 59 and Exercise 5 to the function $g(z) = -if(z)$.

8. Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \ge 1)$. Show in the following way that

$$P(z) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$.

(*a*) Verify that

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z\,z_0^{k-2} + z_0^{k-1}) \qquad (k = 2, 3, \ldots).$$

(*b*) Use the factorization in part (*a*) to show that

$$P(z) - P(z_0) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$, and deduce the desired result from this.

CHAPTER 5

SERIES

This chapter is devoted mainly to series representations of analytic functions. We present theorems that guarantee the existence of such representations, and we develop some facility in manipulating series.

60. CONVERGENCE OF SEQUENCES

An infinite ***sequence*** $z_1, z_2, \ldots, z_n, \ldots$ of complex numbers has a ***limit*** z if, for each positive number ε, there exists a positive integer n_0 such that

$$|z_n - z| < \varepsilon \quad \text{whenever} \quad n > n_0. \tag{1}$$

Geometrically, this means that for sufficiently large values of n, the points z_n lie in any given ε neighborhood of z (Fig. 76). Since we can choose ε as small as we please, it follows that the points z_n become arbitrarily close to z as their subscripts increase. Note that the value of n_0 that is needed will, in general, depend on the value of ε.

A sequence can have at most one limit. That is, a limit z is unique if it exists (Exercise 5, Sec. 61). When the limit z exists, the sequence is said to ***converge*** to z, and we write

$$\lim_{n \to \infty} z_n = z. \tag{2}$$

If a sequence has no limit, it ***diverges***.