Homework. Due: Nov. 24 2015

November 19, 2015

Page: 237 - 238, Q 1 - 4
Q1 Find the residue at $z = 0$ of the function

a $\dfrac{1}{z+z^2}$;

b $z\cos\left(\dfrac{1}{z}\right)$;

c $\dfrac{z-\sin z}{z}$;

d $\dfrac{\cot z}{z^4}$;

e $\dfrac{\sinh z}{z^4(1-z^2)}$.

Ans. a. 1, *b.* $-1/2$*; c.* 0*; d.* $-1/45$*; e.* $7/6$.

Q2 Use Cauchy's residue theorem(Sec.76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

a $\dfrac{\exp(-z)}{z^2}$;

b $\dfrac{\exp(-z)}{(z-1)^2}$;

c $z^2\exp\left(\dfrac{1}{z}\right)$;

d $\dfrac{z+1}{z^2-2z}$

Ans. a. $-2\pi i$*; b.* $-2\pi i/e$*; c.* $\pi i/3$*; d.* $2\pi i$.

Q3 In the example in Sec. 76, two residues were used to evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)}\,dz$$

where C is the positively oriented circle $|z| = 2$. Evaluate this integral once again by using the theorem in Sec. 77 and finding only one residue.

Q4 Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

a $\dfrac{z^5}{1-z^3}$;

b $\dfrac{1}{1+z^2}$;

c $\dfrac{1}{z}$.

Ans. a. $-2\pi i$*; b.* 0*; c.* $2\pi i$*.*

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Q1 In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

a $z\exp\left(\dfrac{1}{z}\right)$;

b $\dfrac{z^2}{1+z}$;

c $\dfrac{\sin z}{z}$;

d $\dfrac{\cos z}{z}$;

e $\dfrac{1}{(2-z)^3}$.

Q2 Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

a $\dfrac{1-\cosh z}{z^3}$;

b $\dfrac{1-\exp(2z)}{z^4}$;

c $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans. a. $m=1,\ B=-1/2$*; b.* $m=3,\ B=-4/3$*; c.* $m=2,\ B=2e^2$*.*

Q3 Suppose that a function f is analytic at z_0, and write $g(z)=f(z)/(z-z_0)$. Show that

a if $f(z_0)\neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

b if $f(z_0)=0$, then z_0 is a removable singular point of g.

suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Page: 264 - 265, Q 2, 4, 9 Use residues to derive the integration formulas in Q2 and Q4

Q2

$$\int_0^\infty \frac{dx}{(x^2+1)^2}=\frac{\pi}{4}.$$

Q4

$$\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$$

Q9 Use a residue and the contour which is the boundary of the sector $0 \le r \le R,\ 0 \le \theta \le 2\pi/3$ and counter-clockwise oriented, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

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Use residues to derive the integration formulas in Q3 and Q5.

Q3

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}\ \ (a>0, b>0).$$

Q5

$$\int_{-\infty}^\infty \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a\ \ (a>0).$$

Use residues to find the Cauchy principal values of the improper integrals in Q8.

Q8

$$\int_{-\infty}^\infty \frac{\sin x\,dx}{x^2+4x+5}.$$

Ans. $-\dfrac{\pi}{e}\sin 2.$

Q12 Follow the steps below to evaluate the *Fresnel integrals*, which are important in diffraction theory:

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

a By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R,\ 0 \le \theta \le \pi/4$ and appealing to the Cauchy-Goursat theorem, show that

$$\int_0^R \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - Re\int_{C_R} e^{iz^2}\,dz$$

and

$$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - Im\int_{C_R} e^{iz^2}\,dz,$$

where C_R is the arc $z = Re^{i\theta}\ (0 \le \theta \le \pi/4)$.

b Show that the value of the integral along the arc C_R in part a tends to zero as R tends to infinity by obtaining the inequality

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi$$

and then referring to the form (2), Sec. 88, of Jordan's inequality.

c Use the results in part a and b, together with the known integration formula

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

to complete the exercise.