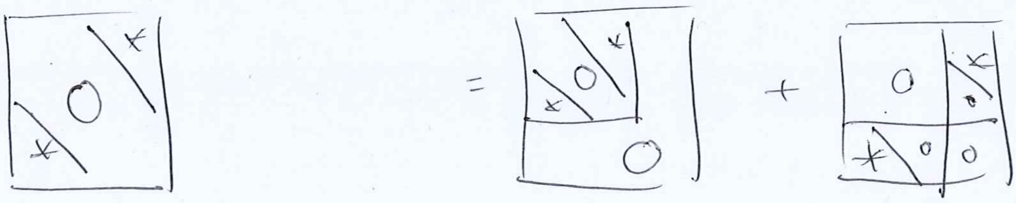
Thm Let $f \in C_{2\pi}$ be $(l+1)$-times differentiable, with $f^{(l+1)} \in L^1(-\pi,\pi)$, $l>0$.

Then for large n
$$\|e_{2q}\|_Q \le \frac{c^q}{((q-1)!)^{2l}} \|e_0\|_Q$$
for some constant c that depends on f & l only.

Pf:
$$a_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ijx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{-ij} f(x)\, de^{-ijx}$$
$$= \frac{-1}{2\pi i j} f(x)\, e^{-ijx}\Big|_{-\pi}^{\pi} + \frac{1}{2\pi i j}\int_{-\pi}^{\pi} f'(x)\, e^{-ijx}\,dx$$
$$= \frac{-1}{2\pi i j}\overbrace{\left\{ f(\pi)\, e^{-ij\pi} - f(-\pi)\, e^{+ij\pi} \right\}}^{0} + \frac{1}{2\pi i j}\int_{-\pi}^{\pi} f'(x)\, e^{-ijx}\,dx$$
$$= \frac{1}{2\pi i j}\int_{-\pi}^{\pi} f'(x)\, e^{-ijx}\,dx$$
$$\therefore\ |a_j| \le \frac{1}{2\pi |j|}\int_{-\pi}^{\pi} |f'(x)|\,dx = \frac{\|f'\|_{L^1(-\pi,\pi)}}{|j|} \le \cdots \le \frac{\|f^{(l+1)}\|_{L^1(-\pi,\pi)}}{|j|^{l+1}} \equiv \hat{c}$$

Hence
$$\sum_{j=k+1}^{n-k-1} |a_j| \le \hat{c} \sum_{j=k+1}^{n-k-1} \frac{1}{|j|^{l+1}} \le \hat{c}\int_k^\infty \frac{dx}{x^{l+1}} \le \frac{\hat{c}}{k^l} \qquad \forall k \ge 1$$

$$B_n = T_n - S_n = W_n^{(k)} + U_n^{(k)}$$

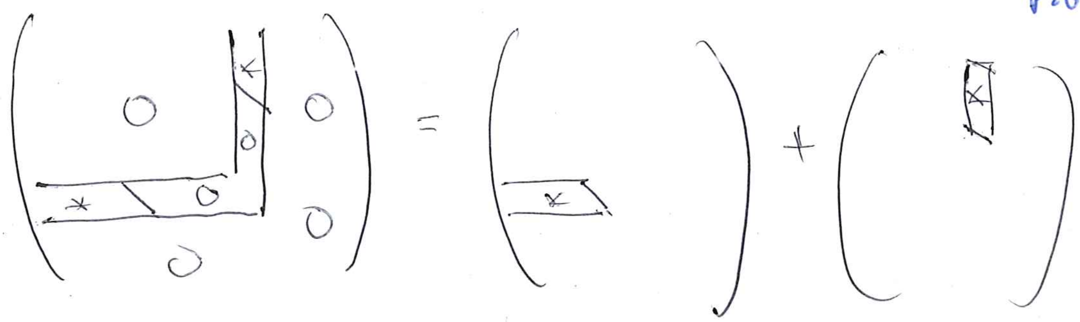


$$\|W_n^{(k)}\|_2 \le \frac{\hat{c}}{k^l} \qquad \mathrm{rank}\,(U_n^{(k)}) \le 2k$$

Recall $\|S_n^{-1}\| \le \frac{1}{f_{min}}$

$$S_n^{-1/2} B_n S_n^{-1/2} = S_n^{-1/2} T_n S_n^{-1/2} - I_n = \tilde{W}_n^{(k)} + \tilde{U}_n^{(k)}$$
$$\|\tilde{W}_n^{(k)}\| \le \frac{\hat{c}}{k^l}\frac{1}{f_{min}} = \frac{\hat{\hat{c}}}{k^l}, \qquad \mathrm{rank}\,(\tilde{U}_n^{(k)}) \le 2k$$

$$W_n^{(k)} - W_n^{(k+1)} =$$



$$= \vec{u}_k \vec{e}_k^{\,*} + \vec{e}_k \vec{u}_k^{\,*} = \frac{1}{2}\left(\vec{w}_k^{+} \vec{w}_k^{+*} - \vec{w}_k^{-} \vec{w}_k^{-*} \right)$$

where $w_k^{\pm} = u_k \pm e_k$.

Thus $\hat{W}_n^{(k)} - \hat{W}_n^{(k+1)} = \frac{1}{2}\left(\hat{w}_k^{+} \hat{w}_k^{+*} - \hat{w}_k^{-} \hat{w}_k^{-*} \right)$

where $\hat{w}_k^{\pm} = S_n^{-1/2} w_k^{\pm}$

$$\therefore \quad \hat{W}_n^{(0)} = \hat{W}_n^{(k)} + \frac{1}{2} \sum_{j=0}^{k-1} \left(\hat{w}_k^{+} \hat{w}_k^{+*} - \hat{w}_k^{-} \hat{w}_k^{-*} \right)$$

$$= \hat{W}_n^{(k)} + V_k^{+} - V_k^{-} \qquad \forall k \geq 1$$

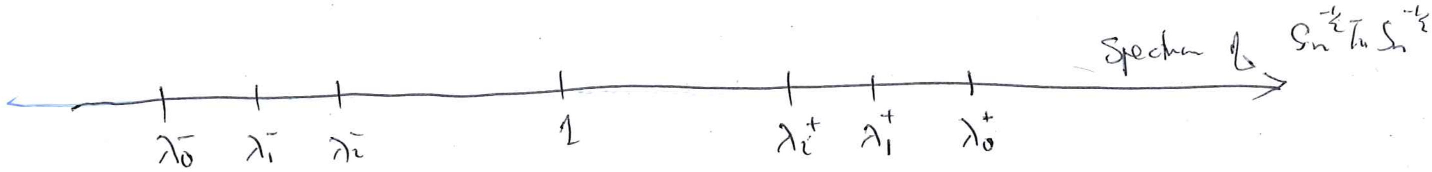
where $V_k^{\pm}$ are positive semi-definite matrices of rank at most k.

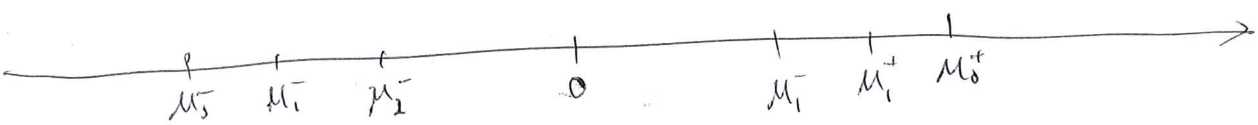
$$S_n^{-1/2} T_n S_n^{-1/2} - I_n = \hat{W}_n^{(k)} + \hat{U}_n^{(k)}$$

$$= \hat{W}_n^{(0)} \leftarrow \mu_i^{\pm}$$

$\lambda_i^{\pm}$

$$= \hat{W}_n^{(k)} + V_k^{+} - V_k^{-} \qquad \forall k \geq 1$$





$$\lambda_i^{\pm} - 1 = \mu_i^{\pm}$$

$$\hat{W}_n^{(0)} = \hat{W}_n^{(1)} + V_1^+ - V_1^-$$

where $\|\hat{W}_n^{(1)}\|_2 \le \frac{\tilde{c}}{1^\ell}$

$$\Rightarrow \quad |\mu_k^\pm| \le \frac{\hat{c}}{1^\ell} \quad \forall k \text{ except the 1st pair. (i.e } \forall k \ge 1)$$

Similarly $\hat{W}_n^{(1)} = \tilde{W}_n^{(2)} + V_2^+ - V_2^-$

where $\|\tilde{W}_n^{(2)}\| \le \frac{\hat{c}}{2^\ell}$

$$\Rightarrow \quad |\mu_k^\pm| \le \frac{\hat{c}}{2^\ell} \quad \forall k \text{ except the 2nd pair. (i.e } \forall k \ge 2)$$

In general $|\mu_k^\pm| \le \frac{\hat{c}}{k^\ell} \quad \forall k$

$$\Rightarrow \quad |\lambda_k^\pm - 1| \le \frac{\hat{c}}{k^\ell} \quad \forall k$$

$$\Rightarrow \quad 1 - \frac{\hat{c}}{k^\ell} \le \lambda_k^- \le \lambda_k^+ \le 1 - \frac{\hat{c}}{k^\ell} \quad \forall k \ge 1 \qquad (1)$$

for $k=0$ $\quad \frac{f_{min}}{f_{max}} \le \lambda_{min}(S_n^{-\frac{1}{2}} T_n S_n^{-\frac{1}{2}}) \le \lambda_0^- \le \lambda_0^+ \le \lambda_{max}(S_n^{-\frac{1}{2}} T_n S_n^{-\frac{1}{2}}) \le \frac{f_{max}}{f_{min}} \qquad (2)$

Now choose

$$P_k(x) = \left(1 - \frac{x}{\lambda_k^+}\right)\left(1 - \frac{x}{\lambda_k^-}\right)$$

$P_k(0) = 1$, & $P_k(\lambda_k^\pm) = 0$, & max of $P_k(x)$ inside $[\lambda_k^-, \lambda_k^+]$ is at $\frac{1}{2}(\lambda_k^+ + \lambda_k^-)$

$$\therefore \max_{x \in [\lambda_k^-, \lambda_k^+]} |P_k(x)| = \frac{(\lambda_k^+ - \lambda_k^-)^2}{4\lambda_k^+ \lambda_k^-} \le \left(\frac{2\hat{c}}{k^\ell}\right)^2 \left(\frac{f_{max}}{2f_{min}}\right)^2 \le \frac{c}{k^{2\ell}} \quad \forall k \ge 1$$

for $k=0$ $\quad \max_{x \in [\lambda_0^-, \lambda_0^+]} |P_0(x)| = \frac{(\lambda_0^+ - \lambda_0^-)^2}{4\lambda_0^+ \lambda_0^-} \le \frac{(f_{max}^2 - f_{min}^2)^2}{4f_{min}}$

$$\therefore \quad |P_{2q}(x)| \le \frac{c^q}{((q-1)!)^{2\ell}} \qquad \#$$

Thm 2.7
(v)

$$A_n = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-(n-1)} \\ a_1 & \ddots & & \\ & & \ddots & \\ a_{n-1} & & a_1 & a_0 \end{pmatrix} \qquad C_n = \begin{pmatrix} c_0 & c_{n-1} & & c_1 \\ c_1 & c_0 & \ddots & \\ \vdots & c_1 & \ddots & c_{n-1} \\ c_{n-1} & & c_1 & c_0 \end{pmatrix}$$

$$\min_{C_n} \| A_n - C_n \|_F^2 = \min_{c_0, c_1, \ldots, c_{n-1}} \| A_n - C_n \|_F^2$$

$c_0 : \; n(c_0 - a_0)^2$

$c_1 : \; (n-1)(c_1 - a_1)^2 + (c_1 - a_{-(n-1)})^2$

$c_2 : \; (n-2)(c_2 - a_2)^2 + 2(c_2 - a_{-(n-2)})^2$

$\vdots$

$c_j = (n-j)(c_j - a_j)^2 + j(c_j - a_{-(n-j)})^2$

$$\frac{\partial}{\partial c_j} = 0 \iff 2(n-j)(c_j - a_j) + 2j(c_j - a_{-(n-j)}) = 0$$

$$\iff n c_j = (n-j) a_j + j a_{-(n-j)}$$

$$\iff c_j = \frac{n-j}{n} a_j - \frac{j}{n} a_{-(n-j)}$$

This formula holds true for general A_n

$$A_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \\ & & & & \\ & & & & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{pmatrix} \qquad c_{n-1} = \frac{a_{12} + a_{23} + \cdots + a_{n-1,n} + a_{n,1}}{n}$$