Thm

$$C_n = F_n^* \Lambda_n F_n$$

$$(F_n)_{jk} = \frac{1}{\sqrt{n}} e^{\frac{2\pi i jk}{n}} \qquad F_n = (\vec{f}_1, \vec{f}_2, \dots, \vec{f}_n) \quad (\vec{f}_j)_k = e^{\frac{2\pi i jk}{n}}$$

Pf: $C_n = c_0 I + c_1 \begin{pmatrix} & 1 & & \\ & & 1 & 0 \\ & & & \ddots \\ & & & & 1 \end{pmatrix} + c_2 \begin{pmatrix} & & 1 & \\ & 0 & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} + \cdots + c_{n-1} \begin{pmatrix} & 1 & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ 1 & & & \end{pmatrix}$

$= c_0 I + c_1 R_1 + c_2 R_2 + \cdots + c_n R_{n-1}$

Claim $FR_j = \Lambda_j F$, in fact $(\vec{f}_0 \dots \vec{f}_{n-1}) R_j = \begin{pmatrix} (\vec{f}_j)_0 & & 0 \\ & \ddots & \\ 0 & & (\vec{f}_j)_{n-1} \end{pmatrix} (\vec{f}_0, \dots, \vec{f}_{n-1})$

Pf: $FR_j = (f_0, \dots, f_{n-1}) \begin{pmatrix} & 1 & \\ 1 & 0 & \ddots \\ & \ddots & & 1 \end{pmatrix} = (\vec{f}_j, \vec{f}_{j+1}, \dots \vec{f}_{n-1}, \vec{f}_1, \dots, \vec{f}_{j-1})$

Consider: $\begin{pmatrix} (f_j)_0 & & 0 \\ & \ddots & \\ 0 & & (f_j)_{n-1} \end{pmatrix} (\vec{f}_0, \dots, \vec{f}_k \dots \vec{f}_{n-1})$

the (j,k)th entry is $\quad (\vec{f}_j)_j \cdot (\vec{f}_k)_j = e^{\frac{2\pi i jj}{n}} \cdot e^{\frac{2\pi i jk}{n}}$

$$= e^{\frac{2\pi i j(j+k)}{n}} = (\vec{f}_{(j+k)})_j$$

$\therefore \begin{pmatrix} (\vec{f}_j)_0 & & \\ & \ddots & \\ & & (\vec{f}_j)_{n-1} \end{pmatrix} (\vec{f}_0, \dots \vec{f}_k, \dots \vec{f}_{n-1}) = (\vec{f}_j, \vec{f}_{j+1}, \dots \dots \vec{f}_{j+n})$

$= (\vec{f}_j, \vec{f}_{j+1}, \dots \vec{f}_{n-1}, \vec{f}_1, \dots, \vec{f}_{j-1}) = \text{R.H.S}$ #

Corollary (i) $F C_n \vec{e}_1 = \Lambda_n F_n e_1 = \Lambda_n \mathbb{1} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = F \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \Rightarrow \{\lambda_i\}_{i=1}^n$ can be obtained in $O(n \log n)$ operations

(ii) $C_n^{-1} \vec{x} = F_n^* \Lambda_n^{-1} F_n \vec{x} = F_n^* \begin{pmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n} \end{pmatrix} F_n \vec{x}$ can be obtained in $O(n \log n)$ operations

Eigenvalue of Discrete Laplacian (Dirichlet BC)

$$T_n \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1^i \\ \vdots \\ u_j^i \\ \vdots \\ u_n^i \end{pmatrix} = \lambda_i \begin{pmatrix} u_1^i \\ \vdots \\ u_j^i \\ \vdots \\ u_n^i \end{pmatrix}$$

look for $u_j^i = \cos(j\theta_i + c)$

$$-\cos((j-1)\theta_i + c) + 2\cos(j\theta_i + c) - \cos((j+1)\theta_i + c) = \lambda_i \cos(j\theta_i + c)$$

$$\Rightarrow \quad -(\cos(j\theta_i + c)\cos\theta_i + \sin(j\theta_i + c)\sin\theta_i) + 2\cos(j\theta_i + c) - (\cos(j\theta_i + c)\cos\theta_i - \sin(j\theta_i + c)\sin\theta_i) = \ldots$$

$$\Rightarrow \quad (2 - 2\cos\theta_i)\cos(j\theta_i + c) = \lambda_i \cos(j\theta_i + c)$$

$$\therefore \quad \lambda_i = 2 - 2\cos\theta_i = 4\sin^2\frac{\theta_i}{2}$$

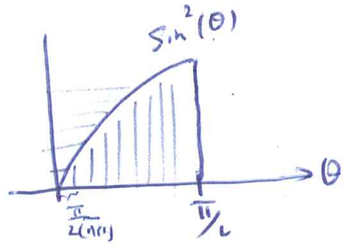
What is θ_i & c.

at $i = 1$: $\quad \cos(0\theta_i + c) = 0 \Rightarrow \cos(c) = 0 \Rightarrow c = \frac{\pi}{2}$

at $i = n$ $\quad \cos((n+1)\theta_i + c) = 0 \Rightarrow \sin((n+1)\theta_i) = 0 \Rightarrow \theta_i = \frac{\pi i}{n+1}$

$$\therefore \quad u_j^i = \cos(j\theta_i + c) = \sin\left(j\frac{\pi i}{n+1}\right)$$

$$\& \quad \lambda_i = 4\sin^2\frac{\pi i}{2(n+1)}, \quad 1 \le i \le n$$



Exercise Eigenvalue of Discrete Laplacian (Neumann B.C)

$$\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & & \ddots & & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

Eigenvalue of Discrete Laplacian (Periodic B.C)

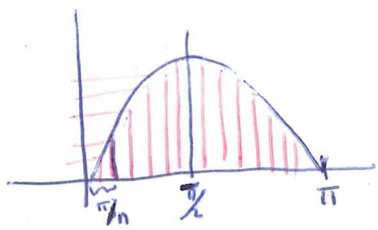
$$\underset{C_n}{\begin{pmatrix} 2 & -1 & & -1 \\ & \ddots & & 0 \\ 0 & -1 & 2 & -1 \\ -1 & & -1 & 2 \end{pmatrix}} \begin{pmatrix} e^{\frac{2\pi i j 0}{n}} \\ e^{\frac{2\pi i j \cdot 1}{n}} \\ e^{\frac{2\pi i j k}{n}} \\ e^{\frac{2\pi i (n-1) j}{n}} \end{pmatrix} =$$

$$= \begin{pmatrix} \vdots \\ -e^{\frac{2\pi i j (k-1)}{n}} + 2 e^{\frac{2\pi i j k}{n}} - e^{\frac{2\pi i j (k+1)}{n}} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ e^{\frac{2\pi i j k}{n}} \\ \vdots \end{pmatrix} \cdot \left(-e^{-\frac{2\pi i j}{n}} + 2 - e^{\frac{2\pi i j}{n}} \right)$$

$$= \left(2 - 2\cos \frac{2\pi j}{n} \right) \begin{pmatrix} \vdots \\ e^{\frac{2\pi i j k}{n}} \\ \vdots \end{pmatrix}$$

$$= 4 \sin^2 \left(\frac{\pi j}{n} \right) \begin{pmatrix} \vdots \\ e^{\frac{2\pi i j k}{n}} \\ \vdots \end{pmatrix}$$

$$\therefore \quad \lambda_j = 4 \sin^2 \left(\frac{\pi j}{n} \right) \qquad 1 \le j \le n$$



For Laplacian

Dirichlet: $\lambda_j(T_n) = 4\sin^2\left(\frac{\pi j}{2(n+1)}\right)$

Circulant: $\lambda_j(C_n) = 4\sin^2\left(\frac{\pi j}{n}\right)$

$$f(x) = e^{-ix} + 2 + e^{ix}$$

$$= 2 - 2\cos x$$

$$= 4\sin^2\left(\frac{x}{2}\right)$$

$f(x)$ has a zero of order 2 at zero.

$$f\left(\frac{2\pi j}{n}\right) = 4\sin^2\left(\frac{2\pi j}{2n}\right) = 4\sin^2\left(\frac{\pi j}{n}\right) = \lambda_j(C_n)$$

<u>Note</u>: (i) $\lambda_j(T_n) = 4\sin^2\left(\frac{\pi j}{2(n+1)}\right) \sim 4\sin^2\left(\frac{\pi j}{n}\right)$

(ii) $\lambda_1(T_n) = 4\sin^2\left(\frac{\pi}{2(n+1)}\right) \sim O(n^{-2})$
