Weyl's Theorem

$A, E \in \mathbb{C}^{n\times n}$

$$\lambda_k(A) + \lambda_1(E) \le \lambda_k(A+E) \le \lambda_k(A) + \lambda_n(E)$$

Pf: Since $U^*(A+E)U$ & $A+E$ has the same eigenvalue.

We can consider $\Lambda + E$, Λ, & E.

$$\lambda_k(\Lambda+E) = \min_{\dim K=k} \max_{K\ni\vec{x}\neq\vec{0}} \frac{\vec{x}^*(\Lambda+E)\vec{x}}{\vec{x}^*\vec{x}}$$

$$= \min_{\dim K=k} \max_{K\ni\vec{x}\neq\vec{0}} \left\{ \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}} + \frac{\vec{x}^*E\vec{x}}{\vec{x}^*\vec{x}} \right\}$$

choose $K_f = \{ \vec{x} \mid (\vec{x})_n = (\vec{x})_{n-1} = \cdots = (\vec{x})_{k+1} = 0 \} \subseteq \mathbb{R}^n$ $\dim K_f = k$

$$\le \max_{K_f\ni\vec{x}\neq\vec{0}} \left\{ \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}} + \frac{\vec{x}^*E\vec{x}}{\vec{x}^*\vec{x}} \right\}$$

$$= \max_{K_f\ni\vec{x}\neq\vec{0}} \left\{ \frac{\sum_{i=1}^{k} \lambda_i(A) x_i^2}{\sum_{i=1}^{k} x_i^2} + \frac{\vec{x}^*E\vec{x}}{\vec{x}^*\vec{x}} \right\}$$

$$\le \lambda_k(A) + \max_{K_f\ni\vec{x}\neq\vec{0}} \frac{\vec{x}^*E\vec{x}}{\vec{x}^*\vec{x}} \le \lambda_k(A) + \lambda_n(E)$$

Since $A = (\Lambda + E) + (-E)$

$$\therefore \lambda_k(A) \le \lambda_k(A+E) + \lambda_n(-E)$$

$$= \lambda_k(A+E) - \lambda_1(E)$$

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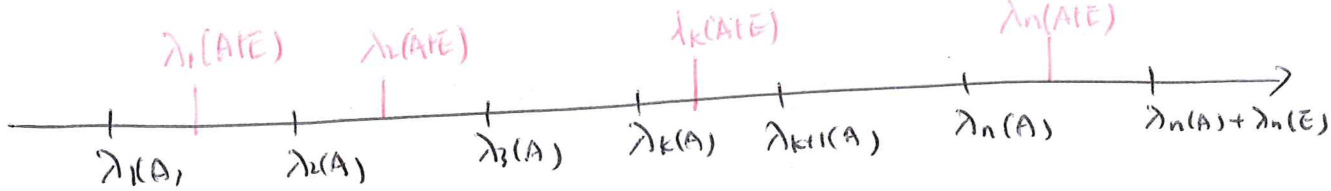
Corollary Let A, E Hermitian, & $\operatorname{rank}(E)=1$, $E \geq 0$

then $\lambda_k(A) \leq \lambda_k(A+E) \leq \lambda_{k+1}(E)$ $\quad 1 \leq k \leq n-1$

$\lambda_n(A) \leq \lambda_n(A+E) \leq \lambda_n(A)+\lambda_n(E)$



interlacing property

Pf: Diagonalizing E by U, $U^*EU = \begin{pmatrix} l & 0 \\ 0 & \ddots & 0 \end{pmatrix} = \Lambda$, $l>0$

Let $U^*AU = B$, since $\lambda_k(A) = \lambda_k(B)$,

w.l.o.g. consider $A + \begin{pmatrix} l & 0 \\ 0 & 0 \end{pmatrix} = A+\Lambda$

By Weyl's thm.

(a) $\lambda_k(A) \leq \lambda_k(A+E)$ $\quad 1 \leq k \leq n$ $\quad (\because \lambda_1(\Lambda)=0)$

(b) $\lambda_n(A+\Lambda) \leq \lambda_n(A)+\lambda_n(\Lambda)$

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n.t.p. $\lambda_k(A+\Lambda) \leq \lambda_{k+1}(A)$ $\quad 1 \leq k \leq n-1$

$$\lambda_{k+1}(A) = \min_{\dim K=k+1} \max_{K \ni \vec{x} \neq \vec{0}} \frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}}$$

Let the min occur at $K_+ = \{\vec{x} \mid \vec{p}_1^*\vec{x}=0, \ldots, \vec{p}_{n-k-1}^*\vec{x}=0\}$ $\quad \dim(K_+)=k+1$

Consider $\bar{K} = K_+ \cap \{(\vec{x})_1 = 0\}$ $\quad \dim \bar{K} = k$

$$\lambda_{k+1}(A) = \max_{K_+ \ni \vec{x} \neq \vec{0}} \frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} \geq \max_{\bar{K} \ni \vec{x} \neq \vec{0}} \frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}}$$

$$= \max_{\bar{K} \ni \vec{x} \neq \vec{0}} \frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}} + \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}}$$

$$\geq \min_{\dim K=k} \max_{K \ni \vec{x} \neq \vec{0}} \left\{ \frac{\vec{x}^*(A+\Lambda)\vec{x}}{\vec{x}^*\vec{x}} \right\}$$

$$= \lambda_k(A+\Lambda)$$ ✱

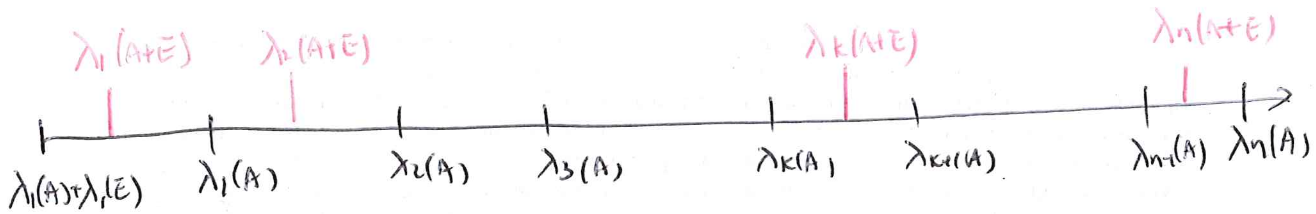
What if one of the $\vec{p}_i$ is $(\vec{x})_1=0$?

Corollary Let A, E Hermitian, & $\operatorname{rank}(E)=1$, $E<0$

then

$$\lambda_{k-1}(A) \le \lambda_k(A+E) \le \lambda_k(A) \qquad 1 \le k \le n$$

$$\lambda_1(A)+\lambda_1(E) \le \lambda_1(A+E) \le \lambda_1(A)$$



Corollary Let A, E Hermitian & $\operatorname{rank}(E)=r$

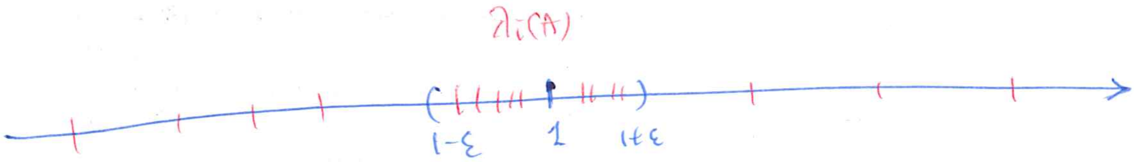
then at most r eigenvalues of $(A+E)$ is outside $[\lambda_1(A), \lambda_n(A)]$

Pf: $\operatorname{rank}(E)=r \Rightarrow E=\sum_{i=1}^{r} L_i$, $\operatorname{rank}(L_i)=1$

Eg: Consider $A_n = I_n + L_n + R_n$ all in $\mathbb{C}^{n\times n}$ Hermitian.

& $\operatorname{rank}(L_n) \le p$, $\|R_n\|_2 \le \varepsilon$. Then $\lambda_i(A) \subseteq [1-\varepsilon, 1+\varepsilon]$

except at most p outliers.



Error in solving linear systems

$$A\vec{x} = \vec{b} \quad , \quad A\hat{x} = \hat{b}$$

Then $$\frac{\|\vec{x} - \hat{x}\|}{\|\vec{x}\|} \le K(A) \frac{\|\vec{b} - \hat{b}\|}{\|\vec{b}\|}$$

pf: $$A(\vec{x} - \hat{x}) = \vec{b} - \hat{b}$$

$$\Rightarrow (\vec{x} - \hat{x}) = A^{-1}(\vec{b} - \hat{b})$$

$$\Rightarrow \|\vec{x} - \hat{x}\| \le \|A^{-1}\| \, \|\vec{b} - \hat{b}\| \qquad (1)$$

$$A\vec{x} = \vec{b}$$

$$\Rightarrow \|\vec{b}\| \le \|A\| \, \|\vec{x}\|$$

$$\Rightarrow \frac{1}{\|\vec{x}\|} \le \frac{\|A\|}{\|\vec{b}\|} \qquad (2)$$

(1) & (2) $$\Rightarrow \frac{\|\vec{x} - \hat{x}\|}{\|\vec{x}\|} \le \|A\| \, \|A^{-1}\| \frac{\|\vec{b} - \hat{b}\|}{\|\vec{b}\|}$$

Note: Matlab 10^{-16} $\Rightarrow \dfrac{\|\vec{b} - \hat{b}\|}{\|\vec{b}\|} = 10^{-16}$

If $K(A) = \|A\| \, \|A^{-1}\| = 10^{8}$

expect $\dfrac{\|\vec{x} - \hat{x}\|}{\|\vec{x}\|} \simeq 10^{-8}$

What if A is also inaccurate?

Thm $A\vec{x} = \vec{b}$, $\hat{A}\hat{x} = \hat{b}$, & $\kappa(A)\dfrac{\|A-\hat{A}\|}{\|A\|} < 1$

Then
$$\frac{\|\vec{x}-\hat{x}\|}{\|\vec{x}\|} \le \frac{\kappa(A)}{1-\kappa(A)\dfrac{\|A-\hat{A}\|}{\|A\|}}\left(\frac{\|A-\hat{A}\|}{\|A\|} + \frac{\|\vec{b}-\hat{b}\|}{\|\vec{b}\|}\right)$$

Pf:-
$$\begin{aligned} A(\vec{x}-\hat{x}) &= A\vec{x} - A\hat{x} \\ &= \vec{b} - A\hat{x} + \hat{A}\hat{x} - \hat{A}\hat{x} \\ &= \vec{b} - (A-\hat{A})\hat{x} - \hat{b} \end{aligned}$$

$$\therefore\ (\vec{x}-\hat{x}) = A^{-1}(\vec{b}-\hat{b}) - A^{-1}(A-\hat{A})\hat{x}$$

$$\frac{\|\vec{x}-\hat{x}\|}{\|\vec{x}\|} \le \|A^{-1}(A-\hat{A})\|\frac{\|\hat{x}\|}{\|\vec{x}\|} + \|A^{-1}\|\frac{\|\vec{b}-\hat{b}\|}{\|b\|}\frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

$$\le \|A^{-1}(A-\hat{A})\|\frac{\|\hat{x}-\vec{x}\|}{\|\vec{x}\|} + \|A^{-1}(A-\hat{A})\|\cancel{\frac{\|\vec{x}\|}{\|\vec{x}\|}} + \|A^{-1}\|\|A\|\frac{\|\vec{b}-\hat{b}\|}{\|b\|}\cancel{\frac{\|\vec{x}\|}{\|\vec{x}\|}}$$

$$\le \|A\|\|A^{-1}\|\frac{\|A-\hat{A}\|}{\|A\|}\frac{\|\hat{x}-\vec{x}\|}{\|\vec{x}\|} + \|A\|\|A^{-1}\|\frac{\|A-\hat{A}\|}{\|A\|} + \kappa(A)\frac{\|\vec{b}-\hat{b}\|}{\|\vec{b}\|}$$

$$\left(1-\kappa(A)\cdot\frac{\|A-\hat{A}\|}{\|A\|}\right)\frac{\|\hat{x}-\vec{x}\|}{\|\vec{x}\|} \le \kappa(A)\left\{\frac{\|A-\hat{A}\|}{\|A\|} + \frac{\|\vec{b}-\hat{b}\|}{\|\vec{b}\|}\right\}$$

$$\therefore\ \frac{\|\vec{x}-\hat{x}\|}{\|\vec{x}\|} \le \frac{\kappa(A)}{1-\kappa(A)\dfrac{\|A-\hat{A}\|}{\|A\|}}\left\{\frac{\|A-\hat{A}\|}{\|A\|} + \frac{\|\vec{b}-\hat{b}\|}{\|\vec{b}\|}\right\} \qquad \#$$

Note for matlab, r.e. $= 10^{-16}$, if $\kappa(A) = 10^{8}$

$$\text{r.e. in } \vec{x} = \frac{10^{8}}{1-10^{8}\cdot 10^{-16}}\cdot\left\{10^{-16}+10^{-16}\right\}$$

$$\simeq 10^{-8} \qquad \#$$

Conjugate Gradient Method

Conjugate Direction: Q Hermitian, $\vec{d}_i, \vec{d}_j \neq \vec{0}$ are conjugate (w.r.t. Q) if

$$\vec{d}_i^* Q \vec{d}_j = 0$$

Thm 1: $Q > 0$, $\{\vec{d}_0, \ldots \vec{d}_k\}$ conjugate $\Rightarrow$ $\{\vec{d}_i\}_{i=0}^k$ linearly independent

pf: Suppose $\sum_{i=0}^{k} \alpha_i d_i = 0$ for some α_i

$$\forall_j \quad 0 = \vec{d}_j^* Q \left(\sum_{i=1}^{k} \alpha_i d_i\right) = \sum_{i=1}^{k} \alpha_i \vec{d}_j^* Q \vec{d}_i = \alpha_j \vec{d}_j^* Q \vec{d}_j$$

$$\because Q > 0, \ \vec{d}_j^* Q \vec{d}_j > 0 \quad \Rightarrow \quad \alpha_j = 0 \quad \forall_j$$

Note: Suppose we solve $Q\vec{x} = \vec{b}$ & solution is $\vec{x}_+$

Suppose we have found n conjugate directions $\{\vec{d}_0, \ldots, \vec{d}_{n-1}\}$ l.i.

$$\vec{x}_+ = \sum_{i=0}^{n-1} \alpha_i \vec{d}_i$$

$$Q\vec{x}_+ = \sum_{i=0}^{n-1} \alpha_i Q\vec{d}_i$$

where $$\vec{d}_j^* Q \vec{x}_+ = \sum_{i=0}^{n-1} \alpha_i \vec{d}_j^* Q \vec{d}_i = \alpha_j \vec{d}_j^* Q \vec{d}_j$$

$$\Rightarrow \quad \alpha_j = \frac{\vec{d}_j^* Q \vec{x}_+}{\vec{d}_j^* Q \vec{d}_j} = \frac{\vec{d}_j^* \vec{b}}{\vec{d}_j^* Q \vec{d}_j}$$

(no information on $\vec{x}_+$ is needed in computing α_j)

$$\vec{x}_+ = \sum_{i=0}^{n-1} \frac{\vec{d}_j^* \vec{b}}{\vec{d}_j^* Q \vec{d}_j} \vec{d}_i$$

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Can we find n conjugate direction easily??

Conjugate Direction Algorithm

Let $\{\vec{d}_0, \dots \vec{d}_{n-1}\}$ be conjugate directions

Given $\vec{x}_0$, define $\vec{x}_{k+1} = \vec{x}_k + \alpha_k \vec{d}_k$

$$\text{where} \quad \alpha_k = -\frac{\vec{g}_k^* \vec{d}_k}{\vec{d}_k^* Q \vec{d}_k} \qquad (*)$$

$$\text{\&} \quad \vec{g}_k = Q\vec{x}_k - \vec{b} \quad \left(= Q(\vec{x}_{k-1} + \alpha_{k-1}\vec{d}_{k-1}) - \vec{b} = \vec{g}_{k-1} + \alpha_{k-1} Q \vec{d}_{k-1} \right)$$

only one matrix-vector multiplication per iteration!

then $\vec{x}_n = Q^{-1}\vec{b} = \vec{x}_+$

pf: $\{\vec{d}_i\}_{i=1}^{n}$ l.i. $\therefore$ $\vec{x}_+ - \vec{x}_0 = \sum_{i=0}^{n-1} \beta_i \vec{d}_i$

$$Q(\vec{x}_+ - \vec{x}_0) = \sum_{i=0}^{n-1} \beta_i Q \vec{d}_i$$

$$\vec{d}_j^* Q(\vec{x}_+ - \vec{x}_0) = \sum_{i=0}^{n-1} \beta_i \vec{d}_j^* Q \vec{d}_i = \beta_j \vec{d}_j^* Q \vec{d}_j \qquad 0 \le j \le n-1$$

$$\therefore \quad \beta_j = \frac{\vec{d}_j^* Q(\vec{x}_+ - \vec{x}_0)}{\vec{d}_j^* Q \vec{d}_j} = \frac{\vec{d}_j^*(\vec{b} - Q\vec{x}_0)}{\vec{d}_j^* Q \vec{d}_j} \qquad (1)$$

$$x_k - x_0 = \alpha_0 \vec{d}_0 + \dots + \alpha_{k-1}\vec{d}_{k-1} \qquad k \ge 1$$

$$Q(x_k - x_0) = \sum_{i=0}^{k-1} \alpha_i Q \vec{d}_i$$

$$\vec{d}_k^* Q(x_k - x_0) = \sum_{i=0}^{k-1} \alpha_i \vec{d}_k^* Q \vec{d}_i = 0$$

$$\Rightarrow \quad \vec{d}_k^* Q x_k = \vec{d}_k^* Q x_0 \qquad (2)$$

$$\therefore (2) \to (1) \quad \Rightarrow \quad \beta_j = \frac{\vec{d}_j^*(b - Qx_j)}{\vec{d}_j^* Q \vec{d}_j}$$

$$= -\frac{\vec{d}_j^* \vec{g}_j}{\vec{d}_j^* Q \vec{d}_j} = \alpha_j \qquad (*)$$

$$\text{\&} \quad \vec{x}_+ = x_0 + \sum_{i=0}^{n-1} \alpha_i \vec{d}_i = \vec{x}_n$$

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Expanding Subspace Theorem

$\{\vec d_i\}_{i=0}^{n-1}$ conjugate direction & $\{\vec x_i\}_{i=0}^{n}$ from Conjugate direction method

Then $\displaystyle \vec x_k = \min_{\vec x \in \vec x_0 + B_k} \left\{ \frac{1}{2} \vec x^{T} Q \vec x - \vec b^{T} x \right\}$

where $B_k = \{\vec d_0, \dots \vec d_{k-1}\}$

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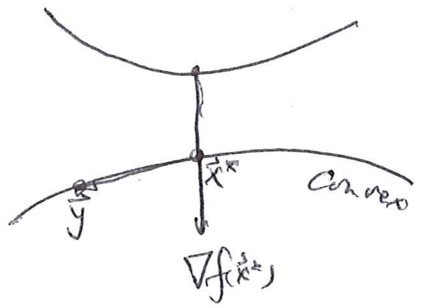
pf: $f(x) \equiv \frac{1}{2} \vec x^{T} Q \vec x - \vec b^{T} \vec x$ strictly convex

$\vec x_0 + B_k$ is affine

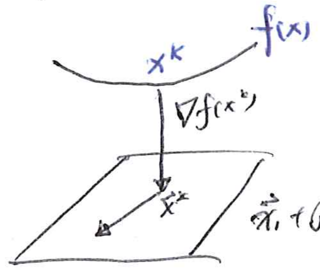


need to prove $\nabla f(\vec x_k) \perp B_k$

$\nabla f(\vec x_k) = Q \vec x_k - \vec b = \vec g_k$

$\nabla f(\vec x^*) (\vec y - \vec x) \geq 0$

By induction, show $\vec g_k \perp B_k \quad \forall k$



$k=0$ $B_k = \phi$, okay

Suppose true for all $i \leq k$, i.e. $\vec g_i \perp B_i$, $i \leq k$

for $k+1$: $\vec g_{k+1} = Q \vec x_{k+1} - \vec b$

$= Q(\vec x_k + \alpha_k \vec d_k) - \vec b$

$= Q \vec x_k + \alpha_k Q \vec d_k - \vec b$

$= \vec g_k + \alpha_k Q \vec d_k$

so for $i = 0, \dots, k-1$ $\quad \vec d_i^{T} \vec g_{k+1} = \underbrace{\vec d_i^{T} \vec g_k}_{\substack{0 \\ \text{(hypothesis)}}} + \alpha_k \underbrace{\vec d_i^{T} Q \vec d_k}_{\substack{0 \\ \text{(conjugate)}}} = 0$

for k $\quad \vec d_k^{T} \vec g_{k+1} = \vec d_k^{T} \vec g_k + \alpha_k \vec d_k^{T} Q \vec d_k = 0$ (by definition of α_k)

$\therefore \quad \vec g_{k+1} \perp \{d_0, \dots, d_k\} = B_{k+1}$

Corollary $\quad \vec g_k^{*} \vec d_i = 0 \quad \forall i = 1, \dots, k-1$