Here we assume that $f \in C_{2\pi}$ has only one zero.

Again w.l.o.g. we assume that the only zero occurs at 0.

$f(\theta) = \sin^{2p}\left(\frac{\theta}{2}\right) g(\theta)$ where $g(\theta) \in C_{2\pi}$, $g(\theta) \gneq 0$.

$$\frac{x^* T_n[f] x}{x^* C_n[K_{m,2r} * f] x} = \underbrace{\frac{x^* T_n[f] x}{x^* T_n[\sin^{2p}(\frac{\theta}{2})] x}}_{1} \underbrace{\frac{x^* T_n[\sin^{2p}(\frac{\theta}{2})] x}{x^* C_n[\sin^{2p}(\frac{\theta}{2})] x}}_{2} \underbrace{\frac{x^* C_n[\sin^{2p}(\frac{\theta}{2})] x}{x^* C_n[f] x}}_{3} \underbrace{\frac{x^* C_n[f] x}{x^* C_n[K_{m,2r} * f] x}}_{4}$$

1 : Similar to the case where we consider band Toeplitz preconditioner

Note that $0 < a_1 \leq \frac{f(\theta)}{\sin^{2p}(\frac{\theta}{2})} = g(\theta) \leq a_2 \quad \forall \theta \in [-\pi, \pi]$

$\therefore \quad a_1 \sin^{2p}(\frac{\theta}{2}) \leq f(\theta) \leq a_2 \sin^{2p}(\frac{\theta}{2})$

$\therefore \quad x^* T_n[f(\theta) - a_1 \sin^{2p}(\frac{\theta}{2})] x \gneq 0$

$\Rightarrow \quad x^* T_n[f(\theta)] x \gneq a_1 x^* T_n[\sin^{2p}(\frac{\theta}{2})] x$

$\Rightarrow \quad 0 < a_1 \leq \frac{x^* T_n[f] x}{x^* T_n[\sin^{2p}(\frac{\theta}{2})] x}$ (Similarly for "$\leq a_2$")

#

For 2 - 4, the circulant matrix C_n is defined as follows:

$$\lambda_j(C_n[h(\theta)]) = \begin{cases} \frac{1}{n^{2p}} & j = 0 \\ h\left(\frac{2\pi j}{n}\right) & \left|\frac{2\pi j}{n}\right| < \frac{\pi}{n} \end{cases}$$

$C_n = F_n^* \Lambda_n F_n$ & $C_n > 0 \quad \forall n$

For ③: since
$$0 < a_1 \le \frac{\lambda_j \left[C_n\left(\sin^2\left(\frac{\theta}{2}\right)\right)\right]}{\lambda_j\left[C_n(f(\theta))\right]} = \frac{\sin^2\left(\frac{2\pi j}{2n}\right)}{f\left(\frac{2\pi j}{2n}\right)} \le a_2 \qquad \left|\frac{2\pi j}{n}\right| < \frac{\pi}{n}$$

$$\frac{\lambda_0\left[C_n\left(\sin^2\frac{\theta}{2}\right)\right]}{\lambda_0\left[C_n(f(\theta))\right]} = 1$$

$$\therefore \quad \frac{x^* C_n\left[\sin^2\left(\frac{\theta}{2}\right)\right] x}{x^* C_n[f(\theta)]\, x} = \frac{u^* F_n C_n\left(\sin^2\left(\frac{\theta}{2}\right)\right) F_n^* u}{u^* F_n C_n(f(\theta)) F_n^* u} = \frac{\sum_{|j|=0}^{n-1} \lambda_j\left(C_n\left(\sin^2\frac{\theta}{2}\right)\right) u_j^2}{\sum_{|j|=0}^{n-1} \lambda_j\left(C_n(f(\theta))\right) u_j^2} \le a_2$$

(Similarly "$\ge a_1$")

For ④: We only need to consider $0 \lneq \left|\frac{2\pi j}{n}\right| < \frac{\pi}{n}$

$$\frac{f(\theta)}{(K_{m,2r} * f)(\theta)} = \underbrace{g(\theta)\cdot\frac{\sin^{2p}\left(\frac{\theta}{2}\right)}{\theta^{2p}}}_{⑤} \cdot \underbrace{\frac{\theta^{2p}}{(K_{m,2r} * \phi^{2p})(\theta)}}_{⑥} \cdot \underbrace{\frac{(K_{m,2r} * \phi^{2p})(\theta)}{\left(K_{m,2r} * \sin^{2p}\left(\frac{\phi}{2}\right)g(\phi)\right)(\theta)}}_{⑦}$$

⑤:
$$b_1 \le g(\theta)\cdot\frac{\sin^{2p}\left(\frac{\theta}{2}\right)}{\theta^{2p}} \le b_2 \qquad \forall\ 0 \lneq \theta = \left|\frac{2\pi j}{n}\right| < \frac{\pi}{n}$$

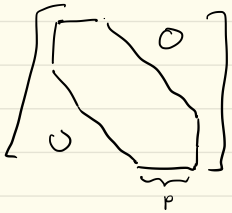
⑦:
$$\left(K_{m,2r} * \sin^{2p}\left(\frac{\phi}{2}\right) g(\phi)\right)(\theta) = \int_{-\pi}^{\pi} K_{m,2r}(\theta-\phi)\,\phi^{2p}\,\frac{\sin^{2p}\left(\frac{\phi}{2}\right)}{\phi^{2p}}\, g(\phi)\, d\phi$$

$$= \underbrace{\frac{\sin^{2p}(\xi)}{\xi^{2p}} g(\xi)}_{\text{positive constant } c} \int_{-\pi}^{\pi} K_{m,2r}(\theta-\phi)\,\phi^{2p}\, d\phi = c\left(K_{m,2r} * \phi^{2p}\right)(\theta)$$

⑥: Theorem 3.4 in paper: $0 < c_1 \le \dfrac{\theta^{2p}}{(K_{m,2r} * \phi^{2p})(\theta)} \le c_2$

② : $\sin^{2p}(\frac{\theta}{2})$ is a trigonometric function of degree p.

$T_n[\sin^{2p}(\frac{\theta}{2})] =$



Recall Strang's preconditioner $\lambda_j(S_n(h(\theta))) = (D_{\frac{n}{2}} * h)(\frac{2\pi j}{n})$

So for $n \gg p$

$$\lambda_j\left(S_n\left(\sin^{2p}\left(\tfrac{\theta}{2}\right)\right)\right) = \sin^{2p}\left(\tfrac{2\pi j}{n}\right) = \lambda_j\left(C_n\left(\sin^{2p}\left(\tfrac{\theta}{2}\right)\right)\right) + R_1$$

where R_1 is rank 1 for $\lambda_0\left(C_n\left(\sin^{2p}\left(\frac{\theta}{2}\right)\right)\right) = \frac{1}{n^{2p}}$.

Note that

$S_n(\sin^{2p}(\frac{\theta}{2})) = T_n(\sin^{2p}(\frac{\theta}{2})) +$

$$\therefore \quad C_n\left(\sin^{2p}\left(\tfrac{\theta}{2}\right)\right) = T_n\left(\sin^{2p}\left(\tfrac{\theta}{2}\right)\right) + R_{2p+1}$$

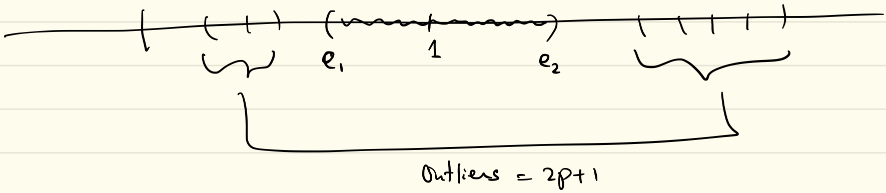
In conclusion:

$$\frac{x^* T_n(f) x}{x^* C_n(K_{m,2r} * f) x} = \alpha(x) + \beta(x) \frac{x^* R_{2p+1} x}{x^* C_n[K_{m,2r} * f] x}$$

$$0 < d_1 \le \alpha(x), \beta(x) \le d_2 \qquad \forall x$$

Corollary

Spectrum of $C_n^{-1}(K_{m,2r} * f) \cdot T_n(f)$



For P.C.G., after $2p+1$ iterations, convergence is like $\left(\frac{\sqrt{\frac{e_2}{e_1}}-1}{\sqrt{\frac{e_2}{e_1}}+1}\right)$:

$$\frac{|||e^{(k+2p+1)}|||}{|||e^{(0)}|||} \leq 2\left(\frac{\sqrt{\frac{e_2}{e_1}}-1}{\sqrt{\frac{e_2}{e_1}}+1}\right)^k$$

#

The Best Circulant Preconditioners for Hermitian Toeplitz Systems

Raymond H. Chan* Andy M. Yip† Michael K. Ng‡

Abstract

In this paper, we propose a new family of circulant preconditioners for ill-conditioned Hermitian Toeplitz systems $A\mathbf{x} = \mathbf{b}$. The preconditioners are constructed by convolving the generating function f of A with the generalized Jackson kernels. For an n-by-n Toeplitz matrix A, the construction of the preconditioners only requires the entries of A and does not require the explicit knowledge of f. When f is a nonnegative continuous function with a zero of order $2p$, the condition number of A is known to grow as $O(n^{2p})$. We show however that our preconditioner is positive definite and the spectrum of the preconditioned matrix is uniformly bounded except for at most $2p+1$ outliers. Moreover the smallest eigenvalue is uniformly bounded away from zero. Hence the conjugate gradient method, when applied to solving the preconditioned system, converges linearly. The total complexity of solving the system is therefore of $O(n \log n)$ operations. In the case when f is positive, we show that the convergence is superlinear. Numerical results are included to illustrate the effectiveness of our new circulant preconditioners.

Key Words. Toeplitz systems, circulant preconditioner, kernel functions, preconditioned conjugate gradient method

AMS(MOS) Subject Classifications. 65F10, 65F15, 65T10

1 Introduction

An n-by-n matrix A_n with entries a_{ij} is said to be Toeplitz if $a_{ij} = a_{i-j}$. Toeplitz systems of the form $A_n\mathbf{x} = \mathbf{b}$ occur in a variety of applications in mathematics and engineering

*E-mail: rchan@math.cuhk.edu.hk. Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong. Research supported in part by Hong Kong Research Grants Council Grant No. CUHK 4207/97P and CUHK DAG Grant No. 2060143.

†E-mail: mhyipa@hkusua.hku.hk. Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong.

‡E-mail: mng@maths.hku.hk. Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong. Research supported in part by HKU 7147/99P and HKU CRCG Grant Nos. 10201939 and 10202720.

[7]. In this paper, we consider the solution of Hermitian positive definite Toeplitz systems. There are a number of specialized fast direct methods for solving such systems in $O(n^2)$ operations, see for instance [22]. Faster methods requiring $O(n \log^2 n)$ operations have also been developed, see [1].

Strang in [21] proposed using the preconditioned conjugate gradient method with circulant matrices as preconditioners for solving Toeplitz systems. The number of operations per iteration is of order $O(n \log n)$ as circulant systems can be solved efficiently by fast Fourier transforms. Several successful circulant preconditioners have been introduced and analyzed; see for instance [11, 5]. In these papers, the given Toeplitz matrix A_n is assumed to be generated by a generating function f, i.e., the diagonals a_j of A_n are given by the Fourier coefficients of f. It was shown that if f is a positive function in the Wiener class (i.e., the Fourier coefficients of f are absolutely summable), then these circulant preconditioned systems converge superlinearly [5]. However, if f has zeros, the corresponding Toeplitz systems will be ill-conditioned. In fact, for the Toeplitz matrices generated by a function with a zero of order $2p$, their condition numbers grow like $O(n^{2p})$, see [19]. Hence the number of iterations required for convergence will increase like $O(n^p)$, see [2, p.24]. Tyrtyshnikov [23] has proved that the Strang [21] and the T. Chan [11] preconditioners both fail in this case.

To tackle this problem, non-circulant type preconditioners have been proposed, see [6, 4, 18, 16]. The basic idea behind these preconditioners is to find a function g that matches the zeros of f. Then the preconditioners are constructed based on the function g. These approaches work when the generating function f is given explicitly, i.e., all Fourier coefficients $\{a_j\}_{j=-\infty}^{\infty}$ of f are available. However, when we are given only a finite n-by-n Toeplitz system, i.e., only $\{a_j\}_{|j|<n}$ are given and the underlying f is unknown, then these preconditioners cannot be constructed. In contrast, most well-known circulant preconditioners, such as the Strang and the T. Chan preconditioners, are defined using only the entries of the given Toeplitz matrix. Di Benedetto in [3] has proved that the condition numbers of the preconditioned matrices by sine transform preconditioners are uniformly bounded. However, the preconditioners themselves may be singular or indefinite in general. Our aim in this paper is to develop a family of positive definite circulant preconditioners that work for ill-conditioned Toeplitz systems and do not require the explicit knowledge of f, i.e., they require only $\{a_j\}_{|j|<n}$ for an n-by-n Toeplitz system.

Our idea is based on the unified approach proposed in Chan and Yeung [9], where they showed that circulant preconditioners can be derived in general by convolving the generating function f with some kernels. For instance, convolving f with the Dirichlet kernel $\mathcal{D}_{\lfloor n/2 \rfloor}$ gives the Strang preconditioner. They proved that for any positive 2π-periodic continuous function f, if $\mathcal{C}_n$ is a kernel such that the convolution product $\mathcal{C}_n * f$ tends to f uniformly on $[-\pi, \pi]$, then the corresponding circulant preconditioned matrix $C_n^{-1} A_n$ will have clustered spectrum. In particular, the conjugate gradient method will converge superlinearly when solving the preconditioned system. This result turns the problem of finding a good preconditioner to the problem of approximating f with $\mathcal{C}_n * f$. Notice that $\mathcal{D}_{\lfloor n/2 \rfloor} * f$, being the partial sum of f, depends solely on the first $\lfloor n/2 \rfloor$

Fourier coefficients $\{a_j\}_{|j|<\lfloor n/2 \rfloor}$ of f. Thus the Strang preconditioner, and similarly for other circulant preconditioners constructed through kernels, does not require the explicitly knowledge of f.

In this paper, we construct our preconditioners by approximating f with the convolution product $\mathcal{K}_{m,2r} * f$ that matches the zeros of f and depends only on $\{a_j\}_{|j|<n}$. Here $\mathcal{K}_{m,2r}$ is chosen to be the generalized Jackson kernels, see [15]. Since $\mathcal{K}_{m,2r}$ are positive kernels, our preconditioners are positive definite for all n. In comparison, the Dirichlet kernel $\mathcal{D}_n$ is not positive and hence the Strang preconditioner is indefinite in general. We will prove that if f has a zero of order $2p$, then $\mathcal{K}_{m,2r} * f$ matches the zero of f when $r > p$. Using this result, we can show that the spectra of the circulant preconditioned matrices are uniformly bounded except for at most $2p+1$ outliers, and that their smallest eigenvalues are bounded uniformly away from zero. It follows that the conjugate gradient method, when applied to solving these circulant preconditioned systems, will converge linearly. Since the cost per iteration is $O(n \log n)$ operations, see [7], the total complexity of solving these ill-conditioned Toeplitz systems is of $O(n \log n)$ operations. In the case when f is positive, we show that the spectra of the preconditioned matrices are clustered around 1 and thus the method converges superlinearly. The case where f has multiple zeros is more involved and will be considered in a future paper.

This paper is an expanded version of the proceedings paper [10] where some of the preliminary results were reported. Recently Potts and Steidl [17] have proposed *skew*-circulant preconditioners for ill-conditioned Toeplitz systems. Their idea is also to use convolution products that match the zeros of f to construct the preconditioners. In particular, they have used the generalized Jackson kernels and the B-spline kernels proposed in [8] in their construction. However, in order to construct the $\{\omega\}$-circulant preconditioners, the position of the zeros of f is required which in general may not be readily available. In contrast, our circulant preconditioners can be constructed without any explicit knowledge of the zeros of f.

The outline of the paper is as follows. In §2, we give an efficient method for computing the eigenvalues of the preconditioners. In §3 we show that $\mathcal{K}_{m,2r} * f$ matches the zeros of f. We then analyze the spectrum of the preconditioned matrices in §4. Numerical results are given in §5 to illustrate the effectiveness of our preconditioners in solving ill-conditioned Toeplitz systems. Concluding remarks are given in §6.

2 Construction of Circulant Preconditioners

Let $\mathbf{C}_{2\pi}$ be the space of all 2π-periodic continuous real-valued functions. The Fourier coefficients of a function f in $\mathbf{C}_{2\pi}$ are given by

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \, d\theta, \quad k = 0, \pm 1, \pm 2, \cdots.$$

Clearly $a_k = \bar{a}_{-k}$ for all k. Let $A_n[f]$ be the n-by-n Hermitian Toeplitz matrix with the (i,j)th entry given by a_{i-j}, $i,j = 0, \ldots, n-1$. We will use $\mathbf{C}_{2\pi}^{+}$ to denote the space of all nonnegative functions in $\mathbf{C}_{2\pi}$ which are not identically zero. We remark that the Toeplitz matrices $A_n[f]$ generated by $f \in \mathbf{C}_{2\pi}^+$ are positive definite for all n, see [6, Lemma 1]. Conversely, if $f \in \mathbf{C}_{2\pi}$ takes both positive and negative values, then $A_n[f]$ will be non-definite. In this paper, we only consider $f \in \mathbf{C}_{2\pi}^+$, i.e., $A_n[f]$ being positive definite Hermitian Toeplitz matrices.

We say that θ_0 is a zero of f of *order* p if $f(\theta_0) = 0$ and p is the smallest positive integer such that $f^{(p)}(\theta_0) \neq 0$ and $f^{(p+1)}(\theta)$ is continuous in a neighborhood of θ_0. By Taylor's theorem,

$$f(\theta) = \frac{f^{(p)}(\theta_0)}{p!}(\theta - \theta_0)^p + O((\theta - \theta_0)^{p+1})$$

for all θ in that neighborhood. Since f is nonnegative, $f^{(p)}(\theta_0) > 0$ and p must be even. We remark that the condition number of $A_n[f]$ generated by such an f grows like $O(n^p)$, see [19]. In this paper, we will consider f having a single zero. The general case where f has multiple zeros is more complicated and will be considered in a future paper.

The systems $A_n[f]\mathbf{x} = \mathbf{b}$ will be solved by the preconditioned conjugate gradient method with circulant preconditioners. It is well known that all n-by-n circulant matrices can be diagonalized by the n-by-n Fourier matrix F_n, see [7]. Therefore, a circulant matrix is uniquely determined by its set of eigenvalues. For a given function f, we define the circulant preconditioner $C_n[f]$ to be the n-by-n circulant matrix with its j-th eigenvalue given by

$$\lambda_j(C_n[f]) = f\left(\frac{2\pi j}{n}\right), \quad 0 \le j < n. \tag{1}$$

We note that $C_n[f] = F_n^* \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) F_n$, see [7]. Hence the matrix-vector multiplication $C_n^{-1}[f]\mathbf{v}$, which is required in each iteration of the preconditioned conjugate gradient method, can be done in $O(n \log n)$ operations by fast Fourier transforms. Clearly if f is a positive function, then $C_n[f]$ is positive definite.

In the following, we will use the generalized Jackson kernel functions

$$\mathcal{K}_{m,2r}(\theta) = \frac{k_{m,2r}}{m^{2r-1}} \left(\frac{\sin(\frac{m\theta}{2})}{\sin(\frac{\theta}{2})}\right)^{2r}, \quad r = 1, 2, \ldots \tag{2}$$

to construct our circulant preconditioners. Here $k_{m,2r}$ is a normalization constant such that $\int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(\theta) d\theta = 1$. It is known that $k_{m,2r}$ is bounded above and below by constants independent of m, see [15, p.57] or (11) below. We note that $\mathcal{K}_{m,2}(\theta)$ is the Fejér kernel.

For any m, the Fejér kernel $\mathcal{K}_{m,2}(\theta)$ can be expressed as

$$\mathcal{K}_{m,2}(\theta) = \sum_{k=-m+1}^{m-1} b_k^{(m,2)} e^{ik\theta},$$

where

$$b_k^{(m,2)} = \frac{m - |k|}{2\pi m}, \quad k = 0, \pm 1, \pm 2, \cdots, \pm(m-1),$$

see for instance [9]. Note that $\int_{-\pi}^{\pi} \mathcal{K}_{m,2}(\theta)d\theta = 2\pi b_0^{(m,2)} = 1$. By (2), we see that $\mathcal{K}_{m,2r}(\theta)$ is the r-th power of $\mathcal{K}_{m,2}(\theta)$ up to a scaling. Hence we have

$$\mathcal{K}_{m,2r}(\theta) = \sum_{k=-r(m-1)}^{r(m-1)} b_k^{(m,2r)} e^{ik\theta}, \tag{3}$$

where the coefficients $b_k^{(m,2r)}$ can be obtained by convolving the vector $(b_{-m+1}^{(m,2)}, \cdots, b_0^{(m,2)}, \cdots, b_{m-1}^{(m,2)})$ with itself for $r-1$ times and this can be done by fast Fourier transforms, see [20, pp.294–296]. Thus the cost of computing the coefficients $\{b_k^{(m,2r)}\}$ for all $|k| \leq r(m-1)$ is of order $O(rm \log m)$ operations. In order to guarantee that $\int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(\theta)d\theta = 1$, we can normalize $b_0^{(m,2r)}$ to $1/(2\pi)$ by dividing all coefficients $b_k^{(m,2r)}$ by $2\pi b_0^{(m,2r)}$.

The convolution product of two arbitrary functions $g = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}$ and $h = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$ in $\mathbf{C}_{2\pi}$ is defined as

$$(g * h)(\theta) \equiv \int_{-\pi}^{\pi} g(t)h(\theta - t)dt = 2\pi \sum_{k=-\infty}^{\infty} b_k c_k e^{ik\theta}. \tag{4}$$

When we are given an n-by-n Toeplitz matrix $A_n[f]$, our proposed circulant preconditioner is $C_n[\mathcal{K}_{m,2r} * f]$, where $m = \lceil n/r \rceil$, i.e.,

$$r(m-1) < n \leq rm. \tag{5}$$

By (3) and (4), since $f = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, the convolution product of $\mathcal{K}_{m,2r} * f$ is given by

$$(\mathcal{K}_{m,2r} * f)(\theta) = 2\pi \sum_{k=-r(m-1)}^{r(m-1)} a_k b_k^{(m,2r)} e^{ik\theta} = \sum_{k=-n+1}^{n-1} d_k e^{ik\theta}, \tag{6}$$

where

$$d_k = \begin{cases} 2\pi a_k b_k^{(m,2r)}, & |k| \leq r(m-1), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{K}_{m,2r} * f$ depends only on a_k for $|k| < n$, i.e., only on the entries of the given n-by-n Toeplitz matrix $A_n[f]$. Notice that by (1), to construct our preconditioner $C_n[\mathcal{K}_{m,2r} * f]$, we only need the values of $\mathcal{K}_{m,2r} * f$ at $2\pi j/n$ for $0 \leq j < n$. By (6), these values can be obtained by taking one fast Fourier transform of length n. Thus the cost of constructing $C_n[\mathcal{K}_{m,2r} * f]$ is of $O(n \log n)$ operations.

We remark that the Strang [21] and the T. Chan circulant preconditioners [11] for $A_n[f]$ are just equal to $C_n[\mathcal{D}_{\lfloor n/2 \rfloor} * f]$ and $C_n[\mathcal{K}_{n,2} * f]$ respectively where $\mathcal{D}_{\lfloor n/2 \rfloor}$ is the Dirichlet kernel and $\mathcal{K}_{n,2}(\theta)$ is the Fejér kernel, see [9].

3 Properties of the Kernel $\mathcal{K}_{m,2r}$

In this section, we study some properties of $\mathcal{K}_{m,2r}$ in order to see how good the approximation of f by $\mathcal{K}_{m,2r} * f$ will be. These properties are useful in the analysis of our circulant preconditioners in §4. First we claim that our preconditioners are positive definite.

Lemma 3.1 *Let $f \in \mathbf{C}_{2\pi}^{+}$. The preconditioner $C_n[\mathcal{K}_{m,2r} * f]$ is positive definite for all positive integers m, n and r.*

Proof: By (2), $\mathcal{K}_{m,2r}(\theta)$ is positive except at $\theta = 2k\pi/m$, $k = \pm 1, \pm 2, \ldots, \pm(n-1)$. Since $f \in \mathbf{C}_{2\pi}^{+}$ is nonnegative and not identically zero, the function

$$(\mathcal{K}_{m,2r} * f)(\theta) \equiv \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) f(\theta - t)\,dt$$

is clearly positive for all $\theta \in [-\pi, \pi]$. Hence by (1), the preconditioners $C_n[\mathcal{K}_{m,2r} * f]$ are positive definite. $\square$

In the following, we will use θ to denote the function θ defined on the whole real line $\mathbb{R}$. For clarity, we will use $\theta_{2\pi}$ to denote the periodic extension of θ on $[-\pi, \pi]$, i.e. $\theta_{2\pi}(\theta) = \tilde{\theta}$ if $\theta = \tilde{\theta} \pmod{2\pi}$ and $\tilde{\theta} \in [-\pi, \pi]$ (cf. Figure 1 below). It is clear that $\theta_{2\pi}^{2p} \in \mathbf{C}_{2\pi}^{+}$ for any integer p. We first show that $\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}$ matches the order of the zero of $\theta_{2\pi}^{2p}$ at $\theta = 0$ if $r > p$.

Lemma 3.2 *Let p and r be positive integers with $r > p$. Then*

$$\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(0) = \left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(0) = \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) t^{2p}\,dt = \frac{c_{p,2r}}{m^{2p}}, \tag{7}$$

where

$$\frac{2^{2p-1}}{2p+1}\left(\frac{2}{\pi}\right)^{4r} \le c_{p,2r} \le 2^{2p+1}\left(\frac{\pi}{2}\right)^{4r}. \tag{8}$$

Proof: The first two equalities in (7) are trivial by the definition of $\theta_{2\pi}$. For the last equality, since $\theta/\pi \le \sin(\theta/2) \le \theta/2$ on $[0, \pi]$, we have by (2)

$$\begin{aligned}
\int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) t^{2p}\,dt &\le \frac{2\pi^{2r} k_{m,2r}}{m^{2r-1}} \int_0^{\pi} \frac{\sin^{2r}\left(\frac{mt}{2}\right)}{t^{2r-2p}}\,dt \\
&= \frac{2^{2p-2r+2}\pi^{2r} k_{m,2r}}{m^{2p}} \int_0^{\frac{m\pi}{2}} \frac{\sin^{2r} u}{u^{2r-2p}}\,du \\
&\le \frac{2^{2p+2} k_{m,2r}}{m^{2p}} \left(\frac{\pi}{2}\right)^{2r} \left\{ \int_0^1 \frac{\sin^{2r} u}{u^{2r-2p}}\,du + \int_1^{\infty} \frac{\sin^{2r} u}{u^{2r-2p}}\,du \right\} \\
&\le \frac{2^{2p+2} k_{m,2r}}{m^{2p}} \left(\frac{\pi}{2}\right)^{2r} \left\{ \int_0^1 u^{2p}\,du + \int_1^{\infty} \frac{1}{u^{2r-2p}}\,du \right\} \\
&\le \frac{2^{2p+3} k_{m,2r}}{m^{2p}} \left(\frac{\pi}{2}\right)^{2r}.
\end{aligned} \tag{9}$$

On the other hand, we also have

$$
\begin{aligned}
\int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) t^{2p} dt &\geq \frac{2^{2r+1} k_{m,2r}}{m^{2r-1}} \int_0^{\pi} \frac{\sin^{2r}\left(\frac{mt}{2}\right)}{t^{2r-2p}} dt \\
&\geq \frac{2^{2p+2} k_{m,2r}}{m^{2p}} \int_0^1 \frac{\sin^{2r} u}{u^{2r-2p}} du \\
&\geq \frac{2^{2p+2} k_{m,2r}}{m^{2p}} \left(\frac{2}{\pi}\right)^{2r} \int_0^1 u^{2p} du \\
&= \frac{2^{2p+2} k_{m,2r}}{(2p+1) m^{2p}} \left(\frac{2}{\pi}\right)^{2r}. \qquad (10)
\end{aligned}
$$

By setting $p = 0$ in (9) and (10), we obtain

$$4\left(\frac{2}{\pi}\right)^{2r} k_{m,2r} \leq 1 = \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) dt \leq 8\left(\frac{\pi}{2}\right)^{2r} k_{m,2r}.$$

Thus

$$\frac{1}{8}\left(\frac{2}{\pi}\right)^{2r} \leq k_{m,2r} \leq \frac{1}{4}\left(\frac{\pi}{2}\right)^{2r}. \qquad (11)$$

Putting (11) back into (9) and (10), we then have (8). $\square$

We remark that using the same arguments as in (10), we can show that

$$(\mathcal{K}_{m,2} * \theta^{2p})(0) \geq O\left(\frac{1}{m}\right), \qquad \forall p \geq 1, \qquad (12)$$

i.e., the T. Chan preconditioner does not match the order of the zeros of θ^{2p} at $\theta = 0$ when $p \geq 1$. We will see in §5 that the T. Chan preconditioner does not work for Toeplitz matrices generated by functions with zeros of order greater than or equal to 2.

Next we estimate $(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p})(\phi)$ for $\phi \neq 0$. In order to do so, we first have to replace the function $\theta_{2\pi}^{2p}$ in the convolution product by θ^{2p} defined on $\mathbb{R}$.

Lemma 3.3 *Let p be a positive integer. Then*

$$\pi^{2p} \leq \frac{\left[\mathcal{K}_{m,2r} * \theta^{2p}(\theta + 2\pi)^{2p}\right](\phi)}{\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi)} \leq \left(\frac{5\pi}{2}\right)^{2p}, \quad \forall \phi \in [-\pi, -\frac{\pi}{2}], \qquad (13)$$

$$1 \leq \frac{\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi)}{\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi)} \leq 3^{2p}, \quad \forall \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}], \qquad (14)$$

and

$$\pi^{2p} \leq \frac{\left(\mathcal{K}_{m,2r} * \theta^{2p}(\theta - 2\pi)^{2p}\right)(\phi)}{\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi)} \leq \left(\frac{5\pi}{2}\right)^{2p}, \quad \forall \phi \in [\frac{\pi}{2}, \pi]. \qquad (15)$$

Proof: To prove (13), we first claim that

$$\pi^{2p} \le \frac{(\phi - t)^{2p}(\phi + 2\pi - t)^{2p}}{(\phi - t)_{2\pi}^{2p}} \le \left(\frac{5\pi}{2}\right)^{2p}, \quad \forall t \in [-\pi, \pi], \phi \in [-\pi, -\frac{\pi}{2}]. \tag{16}$$

By the definition of $(\phi - t)_{2\pi}^{2p}$, we have (see Figure 1)

$$\frac{(\phi - t)^{2p}(\phi + 2\pi - t)^{2p}}{(\phi - t)_{2\pi}^{2p}} = \begin{cases} (\phi + 2\pi - t)^{2p}, & t \in [-\pi, \phi + \pi], \\ (\phi - t)^{2p}, & t \in [\phi + \pi, \pi]. \end{cases}$$

For $t \in [-\pi, \phi + \pi]$ and $\phi \in [-\pi, -\pi/2]$, we have

$$\pi^{2p} = (\phi + 2\pi - (\phi + \pi))^{2p} \le (\phi + 2\pi - t)^{2p} \le (\phi + 3\pi)^{2p} \le \left(\frac{5\pi}{2}\right)^{2p}.$$

For $t \in [\phi + \pi, \pi]$ and $\phi \in [-\pi, -\pi/2]$, we have

$$\pi^{2p} = (\phi - (\phi + \pi))^{2p} \le (\phi - t)^{2p} \le (\phi - \pi)^{2p} \le (2\pi)^{2p}.$$

Thus we have (16).

By (16), we see that

$$\begin{aligned} \left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi) &\equiv \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)(\phi - t)_{2\pi}^{2p} dt \\ &\le \frac{1}{\pi^{2p}} \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)(\phi - t)^{2p}(\phi + 2\pi - t)^{2p} dt \\ &= \frac{1}{\pi^{2p}} \left[\mathcal{K}_{m,2r} * \theta^{2p}(\theta + 2\pi)^{2p}\right](\phi). \end{aligned}$$

Similarly, we also have

$$\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi) \ge \left(\frac{2}{5\pi}\right)^{2p} \left[\mathcal{K}_{m,2r} * \theta^{2p}(\theta + 2\pi)^{2p}\right](\phi).$$

Thus, we have (13).

To prove (14), we just note that

$$1 \le \frac{(\phi - t)^{2p}}{(\phi - t)_{2\pi}^{2p}} \le 3^{2p}, \quad \forall t \in [-\pi, \pi], \phi \in [-\pi/2, \pi/2].$$

As for (15), we have

$$\pi^{2p} \le \frac{(\phi - t)^{2p}(\phi - 2\pi - t)^{2p}}{(\phi - t)_{2\pi}^{2p}} \le \left(\frac{5\pi}{2}\right)^{2p}, \quad \forall t \in [-\pi, \pi], \phi \in [\pi/2, \pi]. \quad \square$$

With Lemmas 3.2 and 3.3, we show that $\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}$ and $\theta_{2\pi}^{2p}$ are essentially the same away from the zero of $\theta_{2\pi}^{2p}$.

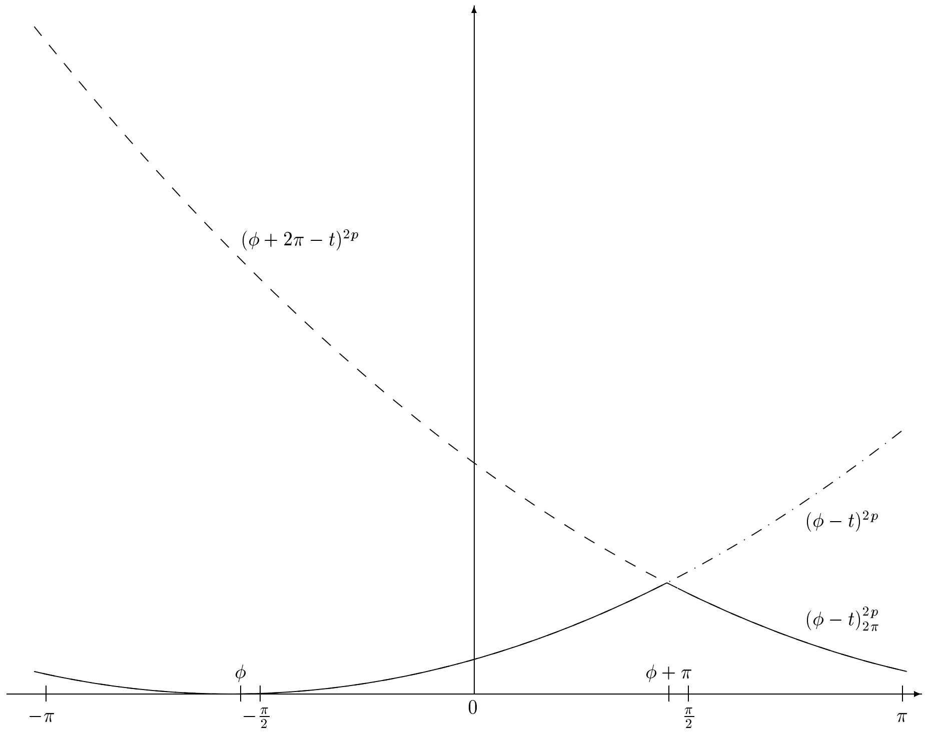


Figure 1: The functions $(\phi - t)_{2\pi}^{2p}$, $(\phi - t)^{2p}$ and $(\phi + 2\pi - t)^{2p}$.

Theorem 3.4 *Let p and r be positive integers with $r > p$ and $m = \lceil n/r \rceil$. Then there exist positive numbers α and β independent of n such that for all sufficiently large n,*

$$\alpha \le \frac{\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} \le \beta, \qquad \forall\ \frac{\pi}{n} \le |\phi| \le \pi. \tag{17}$$

Proof: We see from Lemma 3.3 that for different values of ϕ, $(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p})(\phi)$ can be replaced by different functions. Hence, we proceed the proof for different ranges of values of ϕ.

We first consider $\phi \in [\pi/n, \pi/2]$. By the binomial expansion,

$$\begin{aligned}\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi) &\equiv \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)(\phi - t)^{2p} dt \\ &= \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) \sum_{k=0}^{2p} \binom{2p}{k} \phi^{2p-k}(-t)^k dt.\end{aligned}$$

For odd k, $\int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)t^k dt = 0$. Thus

$$\frac{\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} = \frac{\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi)}{\phi^{2p}} = \sum_{k=0}^{p} \binom{2p}{2k} \phi^{-2k} \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)t^{2k} dt.$$

By (7), we then have

$$\frac{\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} = \sum_{k=0}^{p} \binom{2p}{2k} \frac{c_{k,2r}}{\phi^{2k} m^{2k}}, \tag{18}$$

where by (8), $c_{k,2r}$ are bounded above and below by positive constants independent of m for $k = 0, \ldots p$. Since by (5), $\pi/r \le \pi m/n \le \phi m$, we have

$$c_{0,2r} \le \sum_{k=0}^{p} \binom{2p}{2k} \frac{c_{k,2r}}{\phi^{2k} m^{2k}} \le \sum_{k=0}^{p} \left(\frac{r}{\pi}\right)^{2k} \binom{2p}{2k} c_{k,2r}.$$

Thus by (18),

$$c_{0,2r} \le \frac{\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi)}{\phi_{2\pi}^{2p}} \le \sum_{k=0}^{p} \left(\frac{r}{\pi}\right)^{2k} \binom{2p}{2k} c_{k,2r}.$$

Hence by (14), (17) follows for $\phi \in [\pi/n, \pi/2]$.

The case with $\phi \in [-\pi/2, -\pi/n]$ is similar to the case where $\phi \in [\pi/n, \pi/2]$.

Next we consider the case $\phi \in [\pi/2, \pi]$. Note that

$$\begin{aligned} \left[\mathcal{K}_{m,2r} * \theta^{2p}(\theta - 2\pi)^{2p}\right](\phi) &\equiv \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)(t-\phi)^{2p}(t-\phi+2\pi)^{2p} dt \\ &= \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)\left(\phi^{2p}(2\pi-\phi)^{2p} + q(t)\right) dt \end{aligned}$$

where

$$q(t) = (t-\phi)^{2p}(t-\phi+2\pi)^{2p} - \phi^{2p}(2\pi-\phi)^{2p} \equiv \sum_{j=1}^{4p} q_j t^j$$

is a degree $4p$ polynomial without the constant term. By (7), we have

$$\int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) q(t) dt = \sum_{j=1}^{2p} q_{2j} \frac{c_{2j,2r}}{m^{2j}}.$$

Thus by using the fact that $\int_{-\pi}^{\pi} \mathcal{K}_{m,2r} dt = 1$, we obtain

$$\left[\mathcal{K}_{m,2r} * \theta^{2p}(\theta - 2\pi)^{2p}\right](\phi) = \phi^{2p}(2\pi-\phi)^{2p} + \sum_{j=1}^{2p} q_{2j} \frac{c_{2j,2r}}{m^{2j}}. \tag{19}$$

Since $(\pi/2)^{2p} \leq \phi_{2\pi}^{2p}$ for $\phi \in [\pi/2, \pi]$, we have

$$\begin{aligned}
\frac{\left[\mathcal{K}_{m,2r} * \theta^{2p}(\theta-2\pi)^{2p}\right](\phi)}{\phi_{2\pi}^{2p}} &\leq (2\pi-\phi)^{2p} + \left(\frac{2}{\pi}\right)^{2p} \sum_{j=1}^{2p} |q_{2j}| c_{2j,2r} \\
&\leq \left(\frac{3\pi}{2}\right)^{2p} + \left(\frac{2}{\pi}\right)^{2p} \sum_{j=1}^{2p} |q_{2j}| c_{2j,2r}, \qquad (20)
\end{aligned}$$

which is clearly bounded independent of n. For the lower bound, we use the fact that $\pi^{2p} \geq \phi_{2\pi}^{2p}$ for $\phi \in [\pi/2, \pi]$ in (19), then we have

$$\begin{aligned}
\frac{\left[\mathcal{K}_{m,2r} * \theta^{2p}(\theta-2\pi)^{2p}\right](\phi)}{\phi_{2\pi}^{2p}} &\geq (\phi-2\pi)^{2p} + \frac{1}{\pi^{2p}} \sum_{j=1}^{2p} q_{2j} \frac{c_{2j,2r}}{m^{2j}} \\
&\geq \pi^{2p} + \frac{1}{\pi^{2p}} \sum_{j=1}^{2p} q_{2j} \frac{c_{2j,2r}}{m^{2j}}. \qquad (21)
\end{aligned}$$

Clearly for sufficiently large n (and hence large m), the last expression is bounded uniformly from below say by $\pi^{2p}/2$. Combining (20), (21) and (15), we see that (17) holds for $\phi \in [\pi/2, \pi]$ and for n sufficiently large.

The case where $\phi \in [-\pi, -\pi/2]$ can be proved in a similar way as above. □

Using the fact that

$$\begin{aligned}
[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p}](\phi) &= \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)(\phi-\gamma-t)_{2\pi}^{2p} dt \\
&= (\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p})(\phi-\gamma),
\end{aligned}$$

we obtain the following corollary which deals with functions having a zero at $\gamma \neq 0$.

Corollary 3.5 *Let $\gamma \in [-\pi, \pi]$, p and r be positive integers with $r > p$ and $m = \lceil n/r \rceil$. Then there exist positive numbers α and β, independent of n, such that for all sufficiently large n,*

$$\alpha \leq \frac{\left[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p}\right](\phi)}{(\phi-\gamma)_{2\pi}^{2p}} \leq \beta, \qquad \forall\ \frac{\pi}{n} \leq |\phi-\gamma| \leq \pi.$$

Now we can extend the results in Theorem 3.4 to any functions in $\mathbf{C}_{2\pi}^{+}$ with a single zero of order $2p$.

Theorem 3.6 *Let $f \in \mathbf{C}_{2\pi}^{+}$ and have a zero of order $2p$ at $\gamma \in [-\pi, \pi]$. Let $r > p$ be any integer and $m = \lceil n/r \rceil$. Then there exist positive numbers α and β, independent of n, such that for all sufficiently large n,*

$$\alpha \leq \frac{(\mathcal{K}_{m,2r} * f)(\phi)}{f(\phi)} \leq \beta, \qquad \forall\ \frac{\pi}{n} \leq |\phi-\gamma| \leq \pi.$$

Proof: By the definition of zeros (see §2), $f(\theta) = (\theta-\gamma)_{2\pi}^{2p} g(\theta)$ for some positive continuous function $g(\theta)$ on $[-\pi, \pi]$. Write

$$\frac{(\mathcal{K}_{m,2r} * f)(\phi)}{f(\phi)} = \frac{\left[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p} g(\theta)\right](\phi)}{\left[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p}\right](\phi)} \cdot \frac{\left[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p}\right](\phi)}{(\phi-\gamma)_{2\pi}^{2p}} \cdot \frac{1}{g(\phi)}.$$

Clearly the last factor is uniformly bounded above and below by positive constants. By Corollary 3.5, the same holds for the second factor when $\pi/n \le |\phi-\gamma| \le \pi$. As for the first factor, by the Mean Value Theorem for integrals, there exists a $\zeta \in [-\pi, \pi]$ such that

$$[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p} g(\theta)](\phi) = g(\zeta)[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p}](\phi).$$

Hence

$$0 < g_{\min} \le \frac{\left[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p} g(\theta)\right](\phi)}{\left[\mathcal{K}_{m,2r} * (\theta-\gamma)_{2\pi}^{2p}\right](\phi)} \le g_{\max}, \quad \forall \phi \in [-\pi, \pi],$$

where $g_{\min}$ and $g_{\max}$ are the minimum and maximum of g respectively. Thus the theorem follows. $\square$

So far we have considered only the interval $\pi/n \le |\phi-\gamma| \le \pi$. For $|\phi-\gamma| \le \pi/n$, we now show that the convolution product $\mathcal{K}_{m,2r} * f$ matches the order of the zero of f at the zero of f.

Theorem 3.7 *Let $f \in \mathbf{C}_{2\pi}^{+}$ and have a zero of order $2p$ at $\gamma \in [-\pi, \pi]$. Let $r > p$ be any integer and $m = \lceil n/r \rceil$. Then for any $|\phi-\gamma| \le \pi/n$, we have*

$$(\mathcal{K}_{m,2r} * f)(\phi) = O\left(\frac{1}{n^{2p}}\right).$$

Proof: We first prove the theorem for the function $f(\theta) = \theta_{2\pi}^{2p}$. By the binomial theorem,

$$\begin{aligned}
\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi) &\equiv \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t)(\phi-t)^{2p} dt \\
&= \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) \sum_{j=0}^{2p} \binom{2p}{j} \phi^{2p-j}(-t)^j dt.
\end{aligned}$$

Since $\int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) t^j dt = 0$ for odd j, we have for $|\phi| \le \pi/n$,

$$\begin{aligned}
\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi) &= \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) \sum_{j=0}^{p} \binom{2p}{2j} \phi^{2p-2j} t^{2j} dt \qquad (22) \\
&\le \sum_{j=0}^{p} \binom{2p}{2j} \left(\frac{\pi}{n}\right)^{2p-2j} \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) t^{2j} dt.
\end{aligned}$$

By (7), (8) and (5), we then have

$$\left(\mathcal{K}_{m,2r} * \theta^{2p}\right)(\phi) \le \frac{1}{n^{2p}} \sum_{j=0}^{p} \binom{2p}{2j} r^{2j} \pi^{2p-2j} c_{j,2r} = O\left(\frac{1}{n^{2p}}\right).$$

Hence by (14), $\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi) \le O(1/n^{2p})$. On the other hand, from (22), (8) and (5), we have

$$(\mathcal{K}_{m,2r} * \theta^{2p})(\phi) \ge \int_{-\pi}^{\pi} \mathcal{K}_{m,2r}(t) t^{2p} dt = \frac{c_{p,2r}}{m^{2p}} = O\left(\frac{1}{n^{2p}}\right).$$

Hence by (14) again, $\left(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p}\right)(\phi) \ge O(1/n^{2p})$. Thus the theorem holds for $f(\theta) = \theta_{2\pi}^{2p}$.

In the general case where $f(\theta) = (\theta - \gamma)_{2\pi}^{2p} g(\theta)$ for some positive function $g \in \mathbf{C}_{2\pi}$, by the Mean Value Theorem for integrals, there exists a $\zeta \in [-\pi, \pi]$ such that

$$\begin{aligned} (\mathcal{K}_{m,2r} * f)(\phi) &= [\mathcal{K}_{m,2r} * (\theta - \gamma)_{2\pi}^{2p} g(\theta)](\phi) \\ &= g(\zeta)[\mathcal{K}_{m,2r} * (\theta - \gamma)_{2\pi}^{2p}](\phi) = g(\zeta)(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p})(\phi - \gamma). \end{aligned}$$

Hence

$$g_{\min} \cdot (\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p})(\phi - \gamma) \le (\mathcal{K}_{m,2r} * f)(\phi) \le g_{\max} \cdot (\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p})(\phi - \gamma)$$

for all $\phi \in [-\pi, \pi]$. Here $g_{\min}$ and $g_{\max}$ are the minimum and maximum of g respectively. From the first part of the proof, we already see that $(\mathcal{K}_{m,2r} * \theta_{2\pi}^{2p})(\phi - \gamma)$ is of $O\left(1/n^{2p}\right)$ for all $|\phi - \gamma| \le \pi/n$, hence the theorem follows. $\square$

4 Spectral Properties of the Preconditioned Matrices

4.1 Functions with a Zero

In this subsection, we analyze the spectra of the preconditioned matrices when the generating function has a zero. We will need the following lemma.

Lemma 4.1 *[4, 16] Let $f \in \mathbf{C}_{2\pi}^{+}$. Then $A_n[f]$ is positive definite for all n. Moreover if $g \in \mathbf{C}_{2\pi}^{+}$ is such that $0 < \alpha \le f/g \le \beta$ for some constants α and β, then for all n,*

$$\alpha \le \frac{\mathbf{x}^* A_n[f] \mathbf{x}}{\mathbf{x}^* A_n[g] \mathbf{x}} \le \beta, \qquad \forall \mathbf{x} \ne \mathbf{0}.$$

Next, we have our first main theorem which states that the spectra of the preconditioned matrices are essentially bounded.

Theorem 4.2 *Let $f \in \mathbf{C}_{2\pi}^{+}$ and have a zero of order $2p$ at γ. Let $r > p$ and $m = \lceil n/r \rceil$. Then there exist positive numbers $\alpha < \beta$, independent of n, such that for all sufficiently large n, at most $2p+1$ eigenvalues of $C_n^{-1}[\mathcal{K}_{m,2r} * f]A_n[f]$ are outside the interval $[\alpha, \beta]$.*

Proof: For any function $g \in \mathbf{C}_{2\pi}$, we let $\tilde{C}_n[g]$ to be the n-by-n circulant matrix with the j-th eigenvalue given by

$$\lambda_j(\tilde{C}_n[g]) = \begin{cases} \dfrac{1}{n^{2p}}, & \text{if } \left|\dfrac{2\pi j}{n} - \gamma\right| < \dfrac{\pi}{n} \\ g\left(\dfrac{2\pi j}{n}\right), & \text{otherwise,} \end{cases} \tag{23}$$

for $j = 0, \ldots, n-1$. Since there is at most one j such that $|2\pi j/n - \gamma| < \pi/n$, by (1), $\tilde{C}_n[g] - C_n[g]$ is a matrix of rank at most 1.

By assumption, $f(\theta) = \sin^{2p}((\theta-\gamma)/2)g(\theta)$ for some positive function g in $\mathbf{C}_{2\pi}$. We use the following decomposition of the Rayleigh quotient to prove the theorem:

$$\begin{aligned} \frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}} = {} & \frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* A_n\left[\sin^{2p}\left(\frac{\theta-\gamma}{2}\right)\right]\mathbf{x}} \cdot \frac{\mathbf{x}^* A_n\left[\sin^{2p}\left(\frac{\theta-\gamma}{2}\right)\right]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n\left[\sin^{2p}\left(\frac{\theta-\gamma}{2}\right)\right]\mathbf{x}} \cdot \\ & \cdot \frac{\mathbf{x}^* \tilde{C}_n\left[\sin^{2p}\left(\frac{\theta-\gamma}{2}\right)\right]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n[f]\mathbf{x}} \cdot \frac{\mathbf{x}^* \tilde{C}_n[f]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n[\mathcal{K}_{m,2r} * f]\mathbf{x}} \cdot \\ & \cdot \frac{\mathbf{x}^* \tilde{C}_n[\mathcal{K}_{m,2r} * f]\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}}. \end{aligned} \tag{24}$$

We remark that by Lemma 4.1 and the definitions (1) and (23), all matrices in the factors in the right hand side of (24) are positive definite.

As g is a positive function in $\mathbf{C}_{2\pi}$, by Lemma 4.1, the first factor in the right hand side of (24) is uniformly bounded above and below. Similarly, by (23), the third factor is also uniformly bounded. The eigenvalues of the two circulant matrices in the fourth factor differ only when $|2\pi j/n - \gamma| \geq \pi/n$. But by Theorem 3.6, the ratios of these eigenvalues are all uniformly bounded when n is large. The eigenvalues of the two circulant matrices in the last factor differ only when $|2\pi j/n - \gamma| < \pi/n$. But by Theorem 3.7, their ratios are also uniformly bounded.

It remains to handle the second factor. Define $s_{2p}(\theta) \equiv \sin^{2p}(\frac{\theta-\gamma}{2})$, we have

$$s_{2p}(\theta) = \frac{1}{2^p}[1 - \cos(\theta-\gamma)]^p = \frac{1}{2^p}\left(-\frac{1}{2}e^{i\gamma}e^{-i\theta} + 1 - \frac{1}{2}e^{-i\gamma}e^{i\theta}\right)^p$$

i.e., $s_{2p}(\theta)$ is a p-th degree trigonometric polynomial in θ. Recall that for any function $h(\theta) = \sum_{j=-\infty}^{\infty} b_j e^{ij\theta}$, the convolution product of the Dirichlet kernel $\mathcal{D}_n$ with h is just equal to the nth partial sum of h, i.e., $(\mathcal{D}_n * h)(\theta) = \sum_{j=-n}^{n} b_j e^{ij\theta}$. Hence for $n \geq 2p$, $(\mathcal{D}_{\lfloor n/2 \rfloor} * s_{2p}(\theta))(\phi) = s_{2p}(\phi)$ for all $\phi \in [-\pi, \pi]$.

Since $C_n[\mathcal{D}_{\lfloor n/2 \rfloor} * s_{2p}(\theta)]$ is the Strang preconditioner for $A_n[s_{2p}(\theta)]$, see [9], $C_n[s_{2p}(\theta)]$ will be the Strang preconditioner for $A_n[s_{2p}(\theta)]$ when $n \geq 2p$. As $s_{2p}(\theta)$ is a p-th degree trigonometric polynomial, $A_n[s_{2p}(\theta)]$ is a band Toeplitz matrix with half bandwidth $p+1$. Therefore when $n \geq 2p$, by the definition of the Strang preconditioner,

$$C_n\left[s_{2p}(\theta)\right] = A_n\left[s_{2p}(\theta)\right] + \begin{bmatrix} 0 & 0 & R_p \\ 0 & 0 & 0 \\ R_p^* & 0 & 0 \end{bmatrix}, \tag{25}$$

where R_p is a p-by-p matrix, see [21]. Thus $A_n[s_{2p}(\theta)] = \tilde{C}_n[s_{2p}(\theta)] + R_n$ where the n-by-n matrix R_n is of rank at most $2p+1$.

Putting this back into the numerator of the second factor in (24), we have

$$\begin{aligned} & \frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}} \\ = \quad & \frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* A_n\left[s_{2p}(\theta)\right]\mathbf{x}} \cdot \frac{\mathbf{x}^* \tilde{C}_n\left[s_{2p}(\theta)\right]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n[f]\mathbf{x}} \cdot \frac{\mathbf{x}^* \tilde{C}_n[f]\mathbf{x}}{\mathbf{x}^* \tilde{C}_n[\mathcal{K}_{m,2r} * f]\mathbf{x}} \cdot \frac{\mathbf{x}^* \tilde{C}_n[\mathcal{K}_{m,2r} * f]\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}} \\ & \quad + \frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* A_n\left[s_{2p}(\theta)\right]\mathbf{x}} \cdot \frac{\mathbf{x}^* R_n \mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}}. \end{aligned}$$

Notice that for all sufficiently large n, except for the last factor, all factors above are uniformly bounded below and above by positive constants. We thus have

$$\frac{\mathbf{x}^* A_n[f]\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}} = \alpha(\mathbf{x}) + \beta(\mathbf{x}) \cdot \frac{\mathbf{x}^* R_n \mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}}, \qquad \forall \mathbf{x} \neq \mathbf{0}$$

when n large, where

$$0 < \alpha_{\min} \leq \alpha(\mathbf{x}) \leq \alpha_{\max} < \infty, \quad 0 < \beta_{\min} \leq \beta(\mathbf{x}) \leq \beta_{\max} < \infty.$$

Hence for large n,

$$\frac{\mathbf{x}^*(A_n[f] - \beta_{\max} R_n)\mathbf{x}}{\mathbf{x}^* C_n[\mathcal{K}_{m,2r} * f]\mathbf{x}} \leq \alpha_{\max}, \qquad \forall \mathbf{x} \neq \mathbf{0}.$$

If R_n has q positive eigenvalues, then by Weyl's theorem [13, p.184], at most q eigenvalues of $C_n[\mathcal{K}_{m,2r} * f]^{-1} A_n[f]$ are larger than $\alpha_{\max}$. By using a similar argument, we can prove that at most $2p+1-q$ eigenvalues of $C_n[\mathcal{K}_{m,2r} * f]^{-1} A_n[f]$ are less than $\alpha_{\min}$. Hence the theorem follows. $\square$

Finally we prove that all the eigenvalues of the preconditioned matrices are bounded from below by a positive constant independent of n. Hence the computational cost for solving this class of n-by-n Toeplitz systems will be of $O(n \log n)$ operations.

Theorem 4.3 *Let $f \in \mathbf{C}_{2\pi}^{+}$ and have a zero of order $2p$ at γ. Let $r > p$ and $m = \lceil n/r \rceil$. Then there exists a positive constant c independent of n, such that for all n sufficiently large, all eigenvalues of the preconditioned matrix $C_n^{-1}[\mathcal{K}_{m,2r} * f]A_n[f]$ are larger than c.*

Proof: In view of the proof of Theorem 4.2, it suffices to get a lower bound of the second Rayleigh quotient in the right hand side of (24). Equivalently, we have to get an upper bound of $\rho(A_n^{-1}[s_{2p}(\theta)]\tilde{C}_n[s_{2p}(\theta)])$, where $\rho(\cdot)$ denotes the spectral radius and $s_{2p}(\theta) = \sin^{2p}\left(\frac{\theta-\gamma}{2}\right)$.

We note that by the definition (23), $\tilde{C}_n[s_{2p}(\theta)] = C_n[s_{2p}(\theta)] + E_n$, where E_n is either the zero matrix or is given by

$$F_n^* \mathrm{diag}\left(\cdots, 0, \frac{1}{n^{2p}} - s_{2p}\left(\frac{2\pi j}{n}\right), 0, \cdots\right) F_n$$

for some j such that $|2\pi j/n - \gamma| < \pi/n$. Thus $\|E_n\|_2 = O(1/n^{2p})$.

By Lemma 4.1, $A_n^{-1}[s_{2p}(\theta)]$ is positive definite. Thus the matrix

$$A_n^{-1}[s_{2p}(\theta)]\,\tilde{C}_n[s_{2p}(\theta)]$$

is similar to the symmetric matrix

$$A_n^{-1/2}[s_{2p}(\theta)]\,\tilde{C}_n[s_{2p}(\theta)]\,A_n^{-1/2}[s_{2p}(\theta)].$$

Hence we have

$$\begin{aligned}
& \rho\left(A_n^{-1}[s_{2p}(\theta)]\,\tilde{C}_n[s_{2p}(\theta)]\right) \\
= \;& \rho\left(A_n^{-1/2}[s_{2p}(\theta)]\,\tilde{C}_n[s_{2p}(\theta)]\,A_n^{-1/2}[s_{2p}(\theta)]\right) \\
\leq \;& \rho\left(A_n^{-1/2}[s_{2p}(\theta)]\,C_n[s_{2p}(\theta)]\,A_n^{-1/2}[s_{2p}(\theta)]\right) \\
& \qquad + \rho\left(A_n^{-1/2}[s_{2p}(\theta)]\,E_n A_n^{-1/2}[s_{2p}(\theta)]\right) \\
\leq \;& \rho\left(A_n^{-1}[s_{2p}(\theta)]\,C_n[s_{2p}(\theta)]\right) + \left\|A_n^{-1}[s_{2p}(\theta)]\right\|_2 \|E_n\|_2. \qquad (26)
\end{aligned}$$

Here $\rho(\cdot)$ is the spectral radius of a matrix. By [6, Theorem 1], we have $\left\|A_n^{-1}[s_{2p}(\theta)]\right\|_2 = O(n^{2p})$. Hence the last term in (26) is of $O(1)$.

It remains to estimate the first term in (26). According to (25), we partition $A_n^{-1}[s_{2p}(\theta)]$ as

$$A_n^{-1}[s_{2p}(\theta)] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12}^* & B_{22} & B_{23} \\ B_{13}^* & B_{23}^* & B_{33} \end{bmatrix},$$

where B_{11} and B_{33} are p-by-p matrices. Then by (25),

$$\begin{aligned}
\rho\left(A_n^{-1}\left[s_{2p}(\theta)\right]C_n\left[s_{2p}(\theta)\right]\right) &\leq 1+\rho\left(\left[\begin{array}{ccc} B_{13}R_p^* & 0 & B_{11}R_p \\ B_{23}R_p^* & 0 & B_{12}^*R_p \\ B_{33}R_p^* & 0 & B_{13}^*R_p \end{array}\right]\right) \\
&= 1+\rho\left(\left[\begin{array}{cc} B_{13}R_p^* & B_{11}R_p \\ B_{33}R_p^* & B_{13}^*R_p \end{array}\right]\right), \qquad (27)
\end{aligned}$$

where the last equality follows because the 3-by-3 block matrix in the equation has vanishing central column blocks. In [3, Theorem 4.3], it has been shown that R_p, B_{11}, B_{13} and B_{33} all have bounded ℓ_1-norms and ℓ_∞-norms. Hence using the fact that $\rho(\cdot) \leq \|\cdot\|_2 \leq \{\|\cdot\|_1\|\cdot\|_\infty\}^{1/2}$, we see that (27) is bounded and the theorem follows. $\square$

By combining Theorems 4.2 and 4.3, the number of preconditioned conjugate gradient (PCG) iterations required for convergence is of $O(1)$, see [3]. Since each PCG iteration requires $O(n\log n)$ operations (see [7]) and so is the construction of the preconditioner (see §2), the total complexity of the PCG method for solving Toeplitz systems generated by $f \in \mathbf{C}_{2\pi}^+$ is of $O(n\log n)$ operations.

4.2 Positive Functions

In this subsection, we consider the case where the generating function is strictly positive. We note that the spectrum of $A_n[f]$ is contained in $[f_{\min}, f_{\max}]$, where $f_{\min}$ and $f_{\max}$ are the minimum and maximum values of f, see [6, Lemma 1]. Since $f_{\min} > 0$, $A_n[f]$ is well-conditioned. In [9], it was shown that for such f, the spectrum of $C_n^{-1}[\mathcal{K}_{n,2} * f]A_n[f]$ is clustered around 1 and the PCG method converges superlinearly. Recall that $C_n[\mathcal{K}_{n,2} * f]$ is just the T. Chan circulant preconditioner. In the following, we generalize this result to other generalized Jackson kernels. First, it is easy to show that $0 < f_{\min} \leq (\mathcal{K}_{m,2r} * f)(\phi) \leq f_{\max}$. Thus the whole spectrum of $C_n^{-1}[\mathcal{K}_{m,2r} * f]A_n[f]$ is contained in $[f_{\min}/f_{\max}, f_{\max}/f_{\min}]$, i.e. the preconditioned system is also well-conditioned. We now show that its spectrum is clustered around 1.

Theorem 4.4 *Let $f \in \mathbf{C}_{2\pi}$ be positive. Then the spectrum of $C_n^{-1}[\mathcal{K}_{m,2r} * f]A_n[f]$ is clustered around 1 for sufficiently large n. Here $m = \lceil n/r \rceil$.*

Proof: We first prove that $\mathcal{K}_{m,2r} * f$ converges to f uniformly on $[-\pi, \pi]$. For $\mu > 0$, let $\omega(f,\mu) \equiv \max_{x,|t|\leq\mu} |f(x) - f(x-t)|$ be the modulus of continuity of f. It has the property that

$$\omega(f, \lambda\mu) \leq (\lambda+1)\omega(f,\mu), \quad \forall\, \lambda \geq 0,$$

see [15, p.43].

By the uniform continuity of f, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\omega(f,\delta) < \varepsilon$. Take $n > 1/\delta$, then for all $\phi \in [-\pi, \pi]$, we have

$$|f(\phi) - (\mathcal{K}_{m,2r} * f)(\phi)|$$

$$
\begin{aligned}
&= \left|\int_{-\pi}^{\pi} [\mathcal{K}_{m,2r}(t) f(\phi) - \mathcal{K}_{m,2r}(t) f(\phi - t)]\, dt\right| \\
&\leq \int_{-\pi}^{\pi} |f(\phi) - f(\phi - t)| \mathcal{K}_{m,2r}(t) dt \\
&\leq \int_{-\pi}^{\pi} \omega(f, |t|) \mathcal{K}_{m,2r}(t) dt \\
&= \int_{-\pi}^{\pi} \omega(f, n|t| \cdot \frac{1}{n}) \mathcal{K}_{m,2r}(t) dt \\
&\leq \int_{-\pi}^{\pi} (n|t| + 1) \omega(f, \frac{1}{n}) \mathcal{K}_{m,2r}(t) dt \ = \ \omega(f, \frac{1}{n})\,(c+1) \ \leq \ (c+1)\varepsilon,
\end{aligned}
$$

where $c = 2n \int_0^{\pi} \mathcal{K}_{m,2r}(t) t dt$ is bounded by a constant independent of n (cf. the proof of Lemma 3.2 for $p = 1/2$). Therefore, $\mathcal{K}_{m,2r} * f$ converges uniformly to f. By [9, Theorem 1], the spectrum of $C_n^{-1}[\mathcal{K}_{m,2r} * f] A_n[f]$ is clustered around 1 for sufficiently large n. $\square$

As an immediate consequence, we can conclude that when f is positive and $C_n[\mathcal{K}_{m,2r} * f]$ is used as the preconditioner, the PCG method converges superlinearly, see for instance [5].

5 Numerical Experiments

In this section, we illustrate by numerical examples the effectiveness of the preconditioner $C_n[\mathcal{K}_{m,2r} * f]$ in solving Toeplitz systems. For comparisons, we also test the Strang [21] and the T. Chan [11] circulant preconditioners. In the following, m is set to $\lceil n/r \rceil$.

Example 1: The first set of examples is on *mildly* ill-conditioned Toeplitz systems where the condition numbers of the systems grow like $O(n^{\ell})$ for some $\ell > 0$. They correspond to Toeplitz matrices generated by functions having zeros of order ℓ, see [19]. Because of the ill-conditioning, the conjugate gradient method will converge slowly and the number of iterations required for convergence grows like $O(n^{\ell/2})$ [2, p.24]. However, we will see that using our preconditioner $C_n[\mathcal{K}_{m,2r} * f]$ with $2r > \ell$, the preconditioned system will converge linearly, i.e., the number of iterations required for convergence is independent of n.

We solve Toeplitz systems $A_n[f]\mathbf{x} = \mathbf{b}$ by the preconditioned conjugate gradient method for twelve nonnegative test functions. Since the functions are nonnegative, the $A_n[f]$ so generated are all positive definite. We remark that if f takes negative values, then $A_n[f]$ will be non-definite for large n. As mentioned in §2, the construction of our preconditioners for an n-by-n Toeplitz matrix requires only the n diagonal entries $\{a_j\}_{|j|<n}$ of the given Toeplitz matrix. No explicit knowledge of f is required. In the tests, the right-hand side vectors $\mathbf{b}$ are formed by multiplying random vectors to $A_n[f]$. The initial guess is the zero vector and the stopping criteria is $||\mathbf{r}_q||_2/||\mathbf{r}_0||_2 \leq 10^{-7}$ where $\mathbf{r}_q$ is the residual vector after q iterations.

Tables 1–4 show the numbers of iterations required for convergence for different choices of preconditioners. In the table, I denotes no preconditioner, S is the Strang preconditioner [21], $K_{m,2r}$ are the preconditioners from the generalized Jackson kernel $\mathcal{K}_{m,2r}$ defined in (2) and $T = K_{m,2}$ is the T. Chan preconditioner [11]. Iteration numbers more than 3,000 are denoted by "†". We note that S in general is not positive definite as the Dirichlet kernel $\mathcal{D}_n$ is not positive, see [9]. When some of its eigenvalues are negative, we denote the iteration number by "–" as the PCG method does not apply to non-definite systems and the solution thus obtained may be inaccurate.

The first two test functions in Table 1 are positive functions and therefore correspond to well-conditioned systems. Notice that the iteration number for the non-preconditioned systems tends to a constant when n is large, indicating that the convergence is linear. In this case, we see that all preconditioners work well and the convergence is fast, see Theorem 4.4 and [9].

	θ^4+1						$\|\theta\|^3+0.01$					
n	32	64	128	256	512	1024	32	64	128	256	512	1024
I	33	51	63	69	71	72	45	107	213	288	315	323
S	7	7	7	7	7	7	9	8	8	8	8	8
T	9	8	8	7	7	7	18	19	16	13	10	10
$K_{N,4}$	7	7	7	7	7	7	10	9	9	8	8	8
$K_{N,6}$	7	7	7	7	7	7	10	9	9	8	8	8
$K_{N,8}$	7	7	7	7	7	7	11	9	9	8	8	7

Table 1: Numbers of iterations for well-conditioned systems.

The four test functions in Table 2 are nonnegative functions with single or multiple zeros of order 2 on $[-\pi, \pi]$. Thus the condition numbers of the Toeplitz matrices are growing like $O(n^2)$ and hence the numbers of iterations required for convergence without using any preconditioners is increasing like $O(n)$. We see that for these functions, the number of iterations for convergence using the T. Chan preconditioner increases with n. This is to be expected from the fact. that the order of $\mathcal{K}_{m,2} * \theta^2$ does not match that of θ^2 at $\theta = 0$, see (12). However, we see that $K_{m,4}$, $K_{m,6}$ and $K_{m,8}$ all work very well as predicted from our convergence analysis in §4.

When the order of the zero is 4, like the two test functions in Table 3, the condition number of the Toeplitz matrices will increase like $O(n^4)$ and the matrices will be very ill-conditioned even for moderate n. We see from the table that both the Strang and the T. Chan preconditioners fail (the number of iterations required for convergence is increasing with n). For the T. Chan preconditioner, the failure is also to be expected from the fact that the order of $\mathcal{K}_{m,2} * \theta^4$ does not match that of θ^4 at $\theta = 0$, see (12). As predicted by our theory, $K_{m,6}$ and $K_{m,8}$ still work very well. The numbers of iterations required for convergence are roughly constant independent of n.

In Table 4, we test functions that our theory does not cover. The first two functions are

	θ^2						$(\theta^2-1)^2$					
n	32	64	128	256	512	1024	32	64	128	256	512	1024
I	36	79	170	362	753	1544	53	141	293	547	1113	2213
S	–	–	–	–	–	–	10	–	9	10	8	12
T	12	16	19	23	29	39	18	24	30	27	36	46
$K_{N,4}$	8	9	10	9	9	9	13	13	14	12	13	11
$K_{N,6}$	10	10	10	10	9	9	13	13	14	14	13	13
$K_{N,8}$	9	10	10	10	10	10	13	13	15	15	14	13

	$\theta^2(\pi^2-\theta^2)$						$\theta^2(\pi^4-\theta^4)$					
n	32	64	128	256	512	1024	32	64	128	256	512	1024
I	32	61	116	220	428	835	32	64	128	256	510	1017
S	9	9	9	10	11	12	9	9	9	10	11	12
T	12	14	17	20	26	33	12	15	17	22	27	38
$K_{N,4}$	10	11	11	11	11	11	10	11	11	11	11	11
$K_{N,6}$	10	11	11	11	11	12	10	11	11	11	11	13
$K_{N,8}$	11	12	12	11	12	13	11	12	12	12	12	12

Table 2: Numbers of iterations for functions with order 2 zeros.

	θ^4						$\theta^4(\pi^2-\theta^2)$					
n	32	64	128	256	512	1024	32	64	128	256	512	1024
I	63	209	790	2149	†	†	46	131	410	1084	2600	†
S	–	–	–	–	–	–	–	–	–	–	–	–
T	26	42	71	161	167	247	24	35	58	106	144	196
$K_{N,4}$	15	17	20	24	26	26	15	16	20	22	27	26
$K_{N,6}$	15	16	18	18	17	18	15	16	18	18	18	21
$K_{N,8}$	16	17	19	19	19	20	16	18	19	20	21	23

Table 3: Numbers of iterations for functions with order 4 zeros.

	$\|\theta\|$						$\|\theta\|^3$					
n	32	64	128	256	512	1024	32	64	128	256	512	1024
I	27	40	56	77	110	144	47	128	360	1029	2665	†
S	7	7	8	8	8	8	–	–	–	–	–	–
T	8	9	9	9	10	10	19	28	41	62	98	152
$K_{N,4}$	7	8	8	8	8	9	12	13	13	13	14	15
$K_{N,6}$	8	8	8	8	8	9	12	13	13	12	14	15
$K_{N,8}$	8	8	8	8	8	9	12	14	14	15	14	15

	$\sum_{\|k\|<1024} 1/(\|k\|+1)e^{ik\theta} - 0.3862$						$\sum_{\|k\|<1024} 1/(\|k\|^{0.5}+1)e^{ik\theta} - 0.4325$					
n	32	64	128	256	512	1024	32	64	128	256	512	1024
I	45	112	184	240	296	343	41	92	238	715	1773	†
S	–	–	7	8	8	8	–	–	–	–	–	–
T	15	15	14	12	10	8	14	13	12	17	15	13
$K_{N,4}$	8	8	7	8	8	7	11	10	9	9	9	8
$K_{N,6}$	10	7	8	8	8	8	11	10	9	10	9	9
$K_{N,8}$	9	8	8	8	8	8	11	11	10	10	10	9

Table 4: Numbers of iterations for other functions.

not differentiable at their zeros. The last two functions are functions with slowly decaying Fourier coefficients. We found numerically that the minimum values of $\sum_{|k|<1024} \frac{1}{|k|+1} e^{ik\theta}$ and $\sum_{|k|<1024} \frac{1}{|k|^{0.5}+1} e^{ik\theta}$ are approximately equal to 0.3862 and 0.4325 respectively. Hence the last two test functions are approximately zero at some points in $[-\pi, \pi]$. Table 4 shows that the $K_{m,2r}$ preconditioners still perform better than the Strang and the T. Chan preconditioners.

To further illustrate Theorems 4.2 and 4.3, we give in Figures 2 and 3 the spectra of the preconditioned matrices for all five preconditioners for $f(\theta) = \theta^2$ and θ^4 when $n = 128$. We see that the spectra of the preconditioned matrices for $K_{m,6}$ and $K_{m,8}$ are in a small interval around 1 except for one to two large outliers and that all the eigenvalues are well separated away from 0. We note that the Strang preconditioned matrices in both cases have negative eigenvalues and they are not depicted in the figures.

Example 2: In image restoration, because the blurring is an averaging processing, the resulting matrix is usually *strongly* ill-conditioned in the sense that its condition number grows exponentially with respect to its size n. In contrast, the condition numbers of the mildly ill-conditioned matrices considered in Example 1 are increasing like polynomials of n only. Regularization techniques have been used for some time in mathematics and engineering to treat these strongly ill-conditioned systems. The idea is to restrict the solution in some smooth function spaces [14]. This approach has been adopted in the circulant preconditioned conjugate gradient method and is very successful when applied

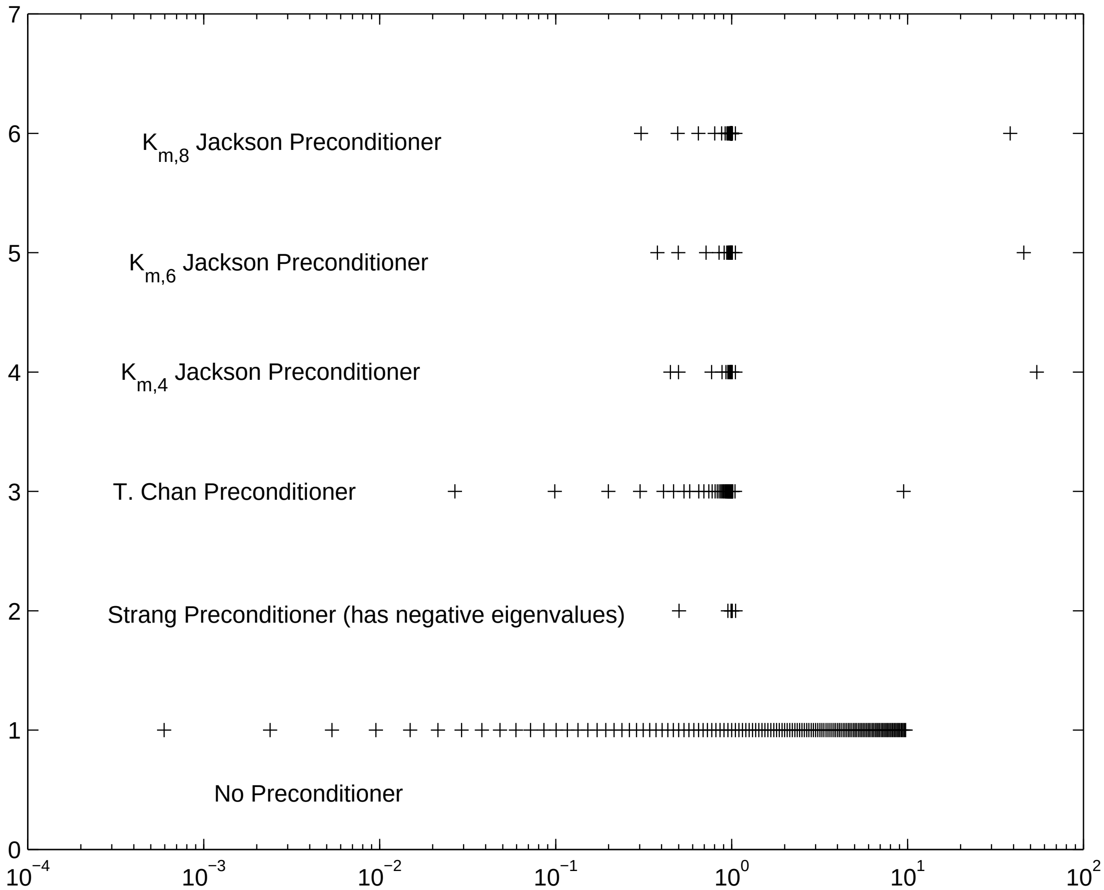


Figure 2: Spectra of preconditioned matrices for $f(\theta) = \theta^2$ when $n = 128$.

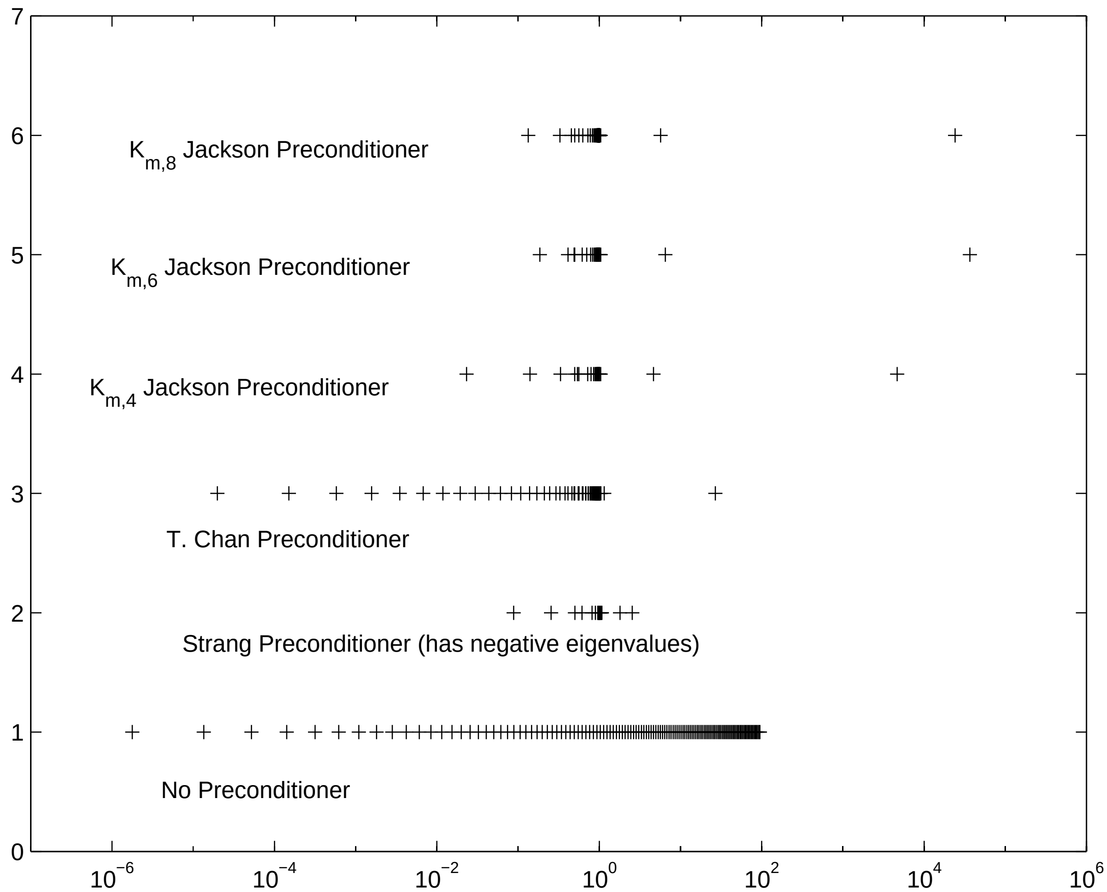


Figure 3: Spectra of preconditioned matrices for $f(\theta) = \theta^4$ when $n = 128$.

to ground-based astronomy [7].

To illustrate the idea, we use a "prototype" image restoration problem given in [12]. Consider a 100-by-100 Toeplitz matrix A with (i, j) entries given by

$$a_{ij} = \begin{cases} 0, & \text{if } |i-j| > 8, \\ \frac{4}{51} g(0.15, \frac{4}{51}(i-j)), & \text{otherwise,} \end{cases}$$

where

$$g(\sigma, \gamma) = \frac{1}{2\sqrt{\pi}\sigma} \exp(-\frac{\gamma^2}{4\sigma^2}).$$

Blurring matrices of this form (called the truncated Gaussian blur) occur in many image restoration contexts and are used to model certain degradations in the recorded image. The condition number of A is approximately 2.3×10^6. Thus if no regularization is used, the result obtained may be very inaccurate, especially if $\mathbf{b}$ is corrupted with noise.

In our experiment, we solve the regularized least squares problem $\min_{\mathbf{x}}\{\|A\mathbf{x} - \mathbf{b}\|_2^2 + \alpha\|\mathbf{x}\|_2^2\}$ as suggested in [12]. The problem is equivalent to the normal equations $(\alpha I + A^2)\mathbf{x} = A\mathbf{b}$ which we solve by the preconditioned conjugate gradient method. We choose the solution vector $\mathbf{x}$ with its entries given by

$$[\mathbf{x}]_i = 0.5g(0.1, -1.1 + \frac{4i}{51}) + g(0.05, -2.8 + \frac{4i}{51}), \quad 1 \le i \le n, \tag{28}$$

see [12], and then we compute $\mathbf{b} = A\mathbf{x}$. A noise vector is added to $\mathbf{b}$ where each component of the noise vector is taken from a normal distribution with mean zero and standard deviation 10^{-3}. The stopping criteria is $||\mathbf{r}_q||_2/||\mathbf{r}_0||_2 \le 10^{-10}$ where $\mathbf{r}_q$ is the residual vector after q iterations.

We choose the optimal regularization parameter α^* such that it minimizes the relative error between the computed solution $\mathbf{x}(\alpha)$ of the normal equations and the original solution $\mathbf{x}$ given in (28), i.e. α^* minimizes $\|\mathbf{x} - \mathbf{x}(\alpha)\|_2/\|\mathbf{x}\|_2$. By trial and error, it is found to be 8×10^{-6} up to one digit of accuracy. The preconditioner we used for the normal equations is of the form $\alpha^* I + C^2$ where C is chosen to be S, T, $K_{m,4}$, $K_{m,6}$, and $K_{m,8}$. The corresponding numbers of iterations required for convergence are equal to $21, 33, 22, 22$, and 23 respectively. The number of iterations without preconditioning is 171. The relative error of the regularized solution is about 3.1×10^{-1}. In contrast, it is about $6.9 \times 10^{+2}$ if no regularization is used. Thus we see that our preconditioners also work for strongly ill-conditioned systems after it is regularized.

6 Concluding Remarks

We remark that even for mildly ill-conditioned matrices with condition number of order $O(n^p)$, if $p > 6$, then the matrix A_n will be very ill-conditioned already for moderate n, say $n = 100$. Thus regularization is also needed in this case. Once the system is regularized, our preconditioner $C_n[\mathcal{K}_{m,8} * f]$ will work even if $p > 6$, cf. Example 2 in §5 for instance. Hence in general, the circulant preconditioner $C_n[\mathcal{K}_{m,8} * f]$ should be able to handle all cases, whether the matrix A_n is well-conditioned, mildly ill-conditioned, or very ill-conditioned but regularized.

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