Courant Fischer Minimax Theorem.

$A \in \mathbb{C}^{n\times n}$ symmetric, $\lambda_1 \le \cdots \le \lambda_n$,

Then
$$\lambda_k = \min_{\dim(K)=k} \max_{\vec{0}\neq\vec{x}\in K} \frac{\vec{x}^*A\vec{x}}{\vec{x}^*\vec{x}}$$

pf:- Since A, & U^TAU have the same set of eigenvalues

W.l.o.g. assume $A = \Lambda$ diagonal matrix.

$\forall K$ with $\dim K = k$, $K = \{\vec{x} \mid \vec{p}_1^*\vec{x}=0, \vec{p}_2^*\vec{x}=0, \ldots, \vec{p}_{n-k}^T\vec{x}=0\}$

Add $(k-1)$ constraints $(\vec{x})_1=0$, $(\vec{x})_2=0, \ldots (\vec{x})_{k-1}=0$.

Then $\exists\ \vec{x}_f \neq \vec{0}$ s.t. it satisfies the $(n-1)$ constraints

$$\vec{x}_f^* = (0, 0, \ldots, 0, x_k, \ldots, x_n) \in K$$

normalize it s.t. $\vec{x}_f^*\vec{x}_f = 1$

$$\frac{\vec{x}_f^*\Lambda\vec{x}_f}{\vec{x}_f^*\vec{x}_f} = \frac{\sum_{j=k}^{n}\lambda_j x_j^2}{1} \ge \lambda_k \sum_{j=k}^{n} x_j^2 = \lambda_k$$

$$\max_{\vec{0}\neq\vec{x}\in K} \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}} \ge \frac{\vec{x}_f^*\Lambda\vec{x}_f}{\vec{x}_f^*\vec{x}_f} = \lambda_k$$

Since K is arbitrary, we have

$$\min_{\dim(K)=k} \max_{\vec{0}\neq\vec{x}\in K} \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}} \ge \lambda_k \qquad (1)$$

Conversely, if we choose $K_f = \{\vec{x} \mid (\vec{x})_n = 0, \ldots, (\vec{x})_{k+1}=0\}$, $\dim(K_f) = k$

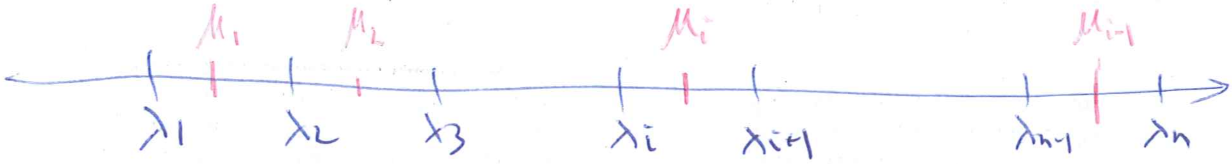
$$\forall \vec{x}\in K_f \qquad \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}} = \frac{\sum_{j=1}^{k}\lambda_j x_j^2}{\sum_{j=1}^{n} x_j^2} \le \lambda_k \frac{\sum_{j=1}^{k} x_j^2}{\sum_{j=1}^{n} x_j^2} \le \lambda_k$$

$$\therefore \max_{\vec{0}\neq\vec{x}\in K_f} \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}} \le \lambda_k \Rightarrow \min_{\dim K=k} \max_{\vec{0}\neq\vec{x}\in K} \frac{\vec{x}^*\Lambda\vec{x}}{\vec{x}^*\vec{x}} \le \lambda_k \qquad (2)$$

Cauchy Interlace Thm

$$A_n = \left[\, a_{ij} \,\right]_{n\times n} \qquad A_{n-1} = \left[\, a_{ij} \,\right]_{(n-1)\times(n-1)}$$

$$\lambda_1 \le \cdots \le \lambda_n \qquad \mu_1 \le \cdots \le \mu_{n-1}$$



pf: $$\mu_i = \min_{\dim(K)=i} \ \max_{K \ni \vec{x}_{n-1} \neq \vec{0}} \ \frac{\vec{x}_{n-1}^{\,*} A_{n-1} \vec{x}_{n-1}}{\vec{x}_{n-1}^{\,*} \vec{x}_{n-1}}$$

Let min occur at $K_+ = \{ \vec{x} \in \mathbb{C}^{n-1} \mid \vec{p}_1^{\,*} \vec{x} = 0, \dots, \vec{p}_{n-1-i}^{\,*} \vec{x} = 0 \} \subseteq \mathbb{C}^{n-1}$

$$\dim K_+ = i$$

$$\mu_i = \max_{K_+ \ni \vec{x}_{n-1} \neq \vec{0}} \ \frac{\vec{x}_{n-1}^{\,*} A_{n-1} \vec{x}_{n-1}}{\vec{x}_{n-1}^{\,*} \vec{x}_{n-1}}$$

define $\hat{K}_+ = \{ \vec{x} \in \mathbb{C}^n \mid \hat{\vec{p}}_1^{\,*} \vec{x} = 0, \hat{\vec{p}}_2^{\,*} \vec{x} = 0, \dots \hat{\vec{p}}_{n-1-i}^{\,*} \vec{x} = 0, (\vec{x})_n = 0 \} \subseteq \mathbb{C}^n$

$\dim \hat{K}_+ = i$, $\hat{p}_i^{\,*} = (\vec{p}_i, 1)$

$$\mu_i = \max_{\hat{K}_+ \ni \vec{x}_n \neq \vec{0}} \ \frac{\vec{x}_n^{\,*} A_n \vec{x}_n}{\vec{x}_n^{\,*} \vec{x}_n} \qquad \dim \hat{K}_+ = i$$

$$\geq \min_{\dim K = i} \ \max_{K \ni \vec{x}_n \neq \vec{0}} \ \frac{\vec{x}_n^{\,*} A_n \vec{x}_n}{\vec{x}_n^{\,*} \vec{x}_n} = \lambda_i \qquad (1)$$

Cauchy's Interlace Theorem (con't)

pf: Let $K_f = \{ \vec{x} \in \mathbb{C}^n \mid \vec{p}_1^{\,T} \vec{x} = 0, \dots, \vec{p}_{n-i}^{\,T} \vec{x} = 0 \} \subseteq \mathbb{C}^n$ $\dim K_f = i$

where

$$\lambda_i = \max_{K_f \ni \vec{x}_n \neq \vec{0}} \frac{\vec{x}_n^{\,*} A_n \vec{x}_n}{\vec{x}_n^{\,*} \vec{x}_n}$$

$$\geq \max_{\substack{K_f \ni \vec{x}_n \neq 0 \\ (x_n)_n = 0}} \frac{\vec{x}_n^{\,*} A_n \vec{x}_n}{\vec{x}_n^{\,*} \vec{x}_n}$$

$$= \max_{\hat{K}_f \ni \vec{x}_{n-1} \neq 0} \frac{\vec{x}_{n-1}^{\,*} A_{n-1} \vec{x}_{n-1}}{\vec{x}_{n-1}^{\,*} \vec{x}_n}$$

where $\hat{K}_f = \{ \vec{x} \in \mathbb{C}^{n-1} \mid \hat{\vec{p}}_1^{\,T} \vec{x} = 0, \dots \hat{\vec{p}}_{n-i}^{\,*} \vec{x} = 0 \} \subseteq \mathbb{C}^{n-1}$ $\dim \hat{K}_f = i-1$

$$\geq \min_{\dim(K) = i-1} \max_{K \ni \vec{x}_{n-1} \neq 0} \frac{\vec{x}_{n-1}^{\,*} A_{n-1} \vec{x}_{n-1}}{\vec{x}_{n-1}^{\,*} \vec{x}_{n-1}} = \mu_{i-1} \qquad (2)$$

Here $\vec{p}_i = (\hat{p}_i, p_i)$, $1 \leq i \leq n-i$

Note: if $\hat{p}_i = 0$ (i.e. $\vec{p}_i = (0, 1)$) then

$$\hat{K}_f = \{ \vec{x} \in \mathbb{C}^{n-1} \mid \hat{\vec{p}}_1^{\,T} \vec{x} = 0, \ \hat{\vec{p}}_{i-1}^{\,T} \vec{x} = 0, \ \hat{\vec{p}}_{i+1}^{\,T} \vec{x} = 0, \dots, \hat{\vec{p}}_{n-1}^{\,T} \vec{x} = 0 \} \subseteq \mathbb{C}^{n-1}$$

$\dim \hat{K}_f = i$

$$\therefore \lambda_i = \max_{\hat{K}_f \ni \vec{x}_{n-1} \neq 0} \frac{\vec{x}_{n-1}^{\,*} A_{n-1} \vec{x}_{n-1}}{\vec{x}_{n-1}^{\,*} \vec{x}_{n-1}}$$

$$\geq \min_{\dim(K) = i} \max_{K \ni \vec{x}_{n-1} \neq 0} \frac{\vec{x}_{n-1}^{\,*} A_{n-1} \vec{x}_{n-1}}{\vec{x}_{n-1}^{\,*} \vec{x}_{n-1}} = \mu_i \geq \mu_{i-1} \qquad (3)$$

$\times$

By (1) & (3) $\lambda_i = \mu_i$ $\times$