MATH 6211 Assignment 2
Due date: Dec. 4

1. Suppose f is a rational function of the form $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are trigonometric polynomials of degrees μ and ν respectively. Then the number of outlying eigenvalues of $(S(T_n))^{-1}T_n$ is exactly equal to $2\max\{\mu,\nu\}$. Hence, the method converges in at most $2\max\{\mu,\nu\}+1$ steps for large n. If, however,

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

is analytic only in a neighborhood of $|z| = 1$, then there exist constants $c > 0$ and $0 \le r < 1$ such that

$$\frac{|||\boldsymbol{e}^{(k+1)}|||}{|||\boldsymbol{e}^{(0)}|||} \le c^k r^{k^2/4+k/2}.$$

2. Suppose f is a Lipschitz function of order ν for $0 < \nu \le 1$, or f has a continuous ν-th derivatives for $\nu \ge 1$. Then there exists a constant $c > 0$ which depends only on f and ν such that for large n,

$$\frac{|||\boldsymbol{e}^{(2k)}|||}{|||\boldsymbol{e}^{(0)}|||} \le \prod_{p=2}^{k} \frac{c\log^2 p}{p^{2\nu}}.$$

3. Let $C_n(T_n)$ be the Tony Chan's preconditioner for some $n\times n$ Toeplitz matrix T_n. As we have learned from lectures, such definition could be extended to an arbitrary $n\times n$ matrices. Show that C_n is a linear operator from $\mathbb{C}^{n\times n}$ to space of circulant matrices and $\|C_n\|_F = 1$.

4. Let f has 0 at θ_0 with order $2l$ and define $\tilde{f}(\theta) = f(\theta+\theta_0)$. Then

$$T_n[\tilde{f}(\theta)] = \Phi_n^* T_n[f(\theta)]\Phi_n,$$

where $\Phi_n = \mathrm{diag}(1, e^{-i\theta_0}, \cdots, e^{-(n-1)i\theta_0})$.

5. Let f has zeroes at $\theta_1, \cdots, \theta_k$ with order $2l_1 < \cdots < 2l_k$. Then $\lambda_1[T_n[f]] = O(n^{-2l_k})$, which implies the conditional number of $T_n[f]$ is $O(n^{2l_k})$.

6. Let $R_n[f]$ be the R. Chan's preconditioner with respect to generating function f. Show that

$$\lambda_j(R_n[f]) = s_{n-1}[f](\frac{2\pi j}{n}), \forall\ 0 \le j < n,$$

where $s_k[f]$ is the partial sum of f up to the kth term.