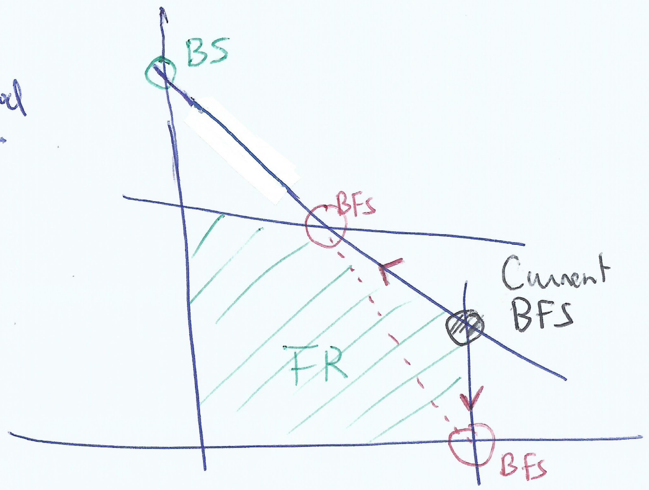
§ 2.3

Simplex Method



Simplex

< n+1 vectors in $\mathbb{R}^n$ >

Direction to go: optimality condition

How far you can go: feasibility condition

Feasibility Condition.

Current BFS $\vec{x} = \begin{pmatrix} \vec{x}_B \\ \vec{0} \end{pmatrix}$

$x_1, \dots, x_m$ basic variables, denoted by x_{B_i}

$x_{m+1} \dots x_n$ non basic variables

$A\vec{x} = \vec{b}$ $\quad A = [\vec{a}_1, \dots \vec{a}_n]$

$\Leftrightarrow [B / R] \begin{bmatrix} x_{B_1} \\ \vdots \\ x_{B_m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{b}$ $\quad B = [\vec{b}_1, \dots, \vec{b}_n]$

$\Leftrightarrow \quad B \vec{x}_B = \vec{b}$

$\Leftrightarrow \quad x_{B_1} \vec{b}_1 + x_{B_2} \vec{b}_2 + \dots + x_{B_m} \vec{b}_m = \vec{b}$ ①

Simplex method: Considering replacing 1 basic variable eg. x_{Br} by 1 non-basic variable x_j

Need to know the relationship between $\vec{b}_1, \dots, \vec{b}_m$ and $\vec{a}_j$, i.e.

$$\Box \vec{b}_1 + \Box \vec{b}_2 + \dots + \Box \vec{b}_m + \Box \vec{a}_j = 0 \quad ??$$

How to get it??

Write

$$[\vec{a}_1, \dots \vec{a}_n] = A = [B \mid R] = B[I \mid B^{-1}R] \equiv BY = B[\vec{y}_1, \dots, \vec{y}_n]$$

i.e. $\vec{a}_j = B\vec{y}_j \quad \forall j$

$$\Leftrightarrow \quad \vec{a}_j = y_{1j}\vec{b}_1 + y_{2j}\vec{b}_2 + \dots + y_{mj}\vec{b}_m \qquad ⋇$$

If we want to replace $\vec{b}_r$ by $\vec{a}_j$, write

$$\vec{b}_r = \frac{1}{y_{rj}}\left\{\vec{a}_j - \sum_{\substack{i=1\\ i\neq r}}^{m} y_{ij}\vec{b}_i\right\} \qquad ②$$

(Here we assume $y_{rj} \neq 0$)

② → ① : $$\sum_{\substack{i=1\\ i\neq r}}^{m}\left(x_{B_i} - x_{B_r}\frac{y_{ij}}{y_{rj}}\right)\vec{b}_i + \frac{x_{Br}}{y_{rj}}\vec{a}_j = \vec{b}$$

$$\Leftrightarrow \quad A\left[\begin{array}{c} x_{B_1} - x_{B_r}\frac{y_{ij}}{y_{rj}} \\ * \\ \vdots \\ * \\ 0 \\ * \\ \vdots \\ * \\ x_{B_m} - x_{B_r}\frac{y_{mj}}{y_{rj}} \\ \hdashline 0 \\ \vdots \\ 0 \\ x_{Br}/y_{rj} \\ 0 \\ \vdots \\ 0 \end{array}\right]\begin{array}{l} \\ \\ \\ \\ \leftarrow r\text{th} \\ \\ \\ \\ \leftarrow m\text{th} \\ \\ \\ \\ \leftarrow j\text{th} \\ \\ \\ \leftarrow n\text{th} \end{array} = \vec{b} \quad \Leftrightarrow \quad A\hat{x}_B = \vec{b} \qquad ③$$

x_r = leaving variable

x_j = entering variable

Thus x_r becomes nonbasic & x_j becomes basic.

But will $\hat{x}_B$ feasible ??

Use Choice 1 in Thm 2.2.

Choose y_{rj} s.t. (i) $y_{rj} > 0$

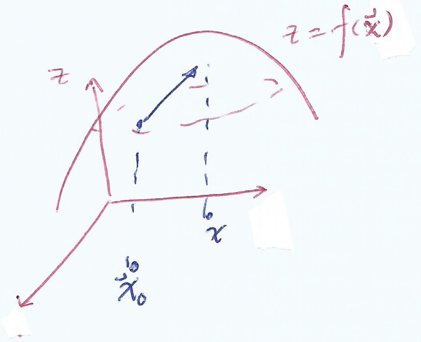
(ii) $\dfrac{x_{Br}}{y_{rj}} = \min\limits_{1 \le i \le m} \left\{ \dfrac{x_{Bi}}{y_{ij}} \;\middle|\; y_{ij} > 0 \right\}$

Choose r with such property "feasibility test"

x_r is the leaving variable.

then easy to verify $\hat{x}_B \ge 0$

Optimality Condition to determine which j to enter.



$$f(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0) + \cdots$$

$$f(\vec{x}) = \vec{c}^T \vec{x} \quad \text{linear} \quad \& \quad \nabla f(x) = \vec{c}^T$$

$$\therefore \quad \vec{c}^T \vec{x} = \vec{c}^T \vec{x}_0 + \vec{c}^T (\vec{x} - \vec{x}_0) \qquad ④$$

If $\nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0) > 0$, then $f(\vec{x}) > f(\vec{x}_0)$.

i.e. $\vec{c}^T(\vec{x} - \vec{x}_0) > 0$, then objective function value increases in the next step.

Here current solution is $\vec{x} = \begin{pmatrix} \vec{x}_B \\ 0 \end{pmatrix}$,

new solution is $\hat{x}_B$ in ③

$$\hat{x}_B - \vec{x} = \left(\begin{array}{c} -x_{B_r} \dfrac{y_{1j}}{y_{rj}} \\ * \\ * \\ \vdots \\ -x_{Br} \\ -x_{B_{r+1}} \dfrac{y_{r+1\,j}}{y_{rj}} \\ \vdots \\ -x_{B_r} \dfrac{y_{mj}}{y_{rj}} \\ \hline 0 \\ \vdots \\ 0 \\ x_{Br}/y_{rj} \\ 0 \\ \vdots \\ 0 \end{array} \right) \begin{array}{l} \\ \\ \\ \\ \leftarrow r\text{th} \\ \\ \\ \\ \\ \\ \\ \leftarrow j\text{th} \\ \\ \\ \\ \end{array}$$

Write $\vec{c}^T = (\underbrace{c_1, \dots, c_m}_{\vec{c}_B^T}, c_{m+1}, \dots, c_n) = (\vec{c}_B^T, *) = (c_{B_1}, c_{B_2}, \dots c_{B_m}, * \cdots *)$

$$\vec{c}^T(\hat{X}_B - \vec{X}) = -\sum_{\substack{i=1\\ i\neq r}}^{m} c_{B_i} X_{Br} \frac{y_{ij}}{y_{rj}} - X_{Br}\underset{\uparrow\,\left(\frac{y_{rj}}{y_{rj}}\right)}{c_{Br}} + c_j \frac{X_{Br}}{y_{rj}}$$

$$= -\frac{X_{Br}}{y_{rj}} \sum_{i=1}^{m} c_{B_i} y_{ij} + \frac{X_{Br}}{y_{rj}} c_j$$

$$= \frac{X_{Br}}{y_{rj}} \left\{ c_j - \underbrace{\sum_{i=1}^{m} c_{B_i} y_{ij}}_{z_j} \right\}$$

$$[z_1, \dots z_n] = [c_{B_1} \cdots c_{B_m}] \cdot Y$$

$$\vec{z}^T = \vec{c}_m^T Y \qquad ⑤$$

$$= \frac{X_{Br}}{y_{rj}} \{ \underbrace{c_j - z_j}_{\text{reduced cost coefficients}} \}$$

Thus if $c_j - z_j \gneq 0$ then $\underset{\substack{\| \\ \text{next} \\ \text{objective} \\ \text{value}}}{\vec{c}^T\hat{X}_B} \gneq \underset{\substack{\| \\ \text{current} \\ \text{objective} \\ \text{value}}}{\vec{c}^T\vec{X}}$

(remember $X_{Br} > 0$, $y_{rj} > 0$)

$$\boxed{\text{Choose } j \text{ s.t. } c_j - z_j > 0 \text{ as entering variable}}$$

"Optimality test" to choose the entering variable x_j

By ④ at $\hat{X}_B$, the new objective value is

$$\vec{c}^T\hat{X}_B = \vec{c}^T\vec{X} + \vec{c}^T(\hat{X}_B - \vec{X})$$

$$= \underset{\substack{\uparrow \\ \text{current objective} \\ \text{value}}}{\vec{c}^T\vec{X}} + \boxed{\frac{X_{Br}}{y_{rj}}(c_j - z_j)}$$

Amount of increase in the next step.

Thm: If $c_j - z_j \le 0$ $\forall j$, we are at the maximum

Pf: $\forall \vec{u} \in FR$ $\quad \begin{cases} A\vec{u} = \vec{b} \\ \vec{u} \ge 0 \end{cases}$

$$\begin{aligned}
\vec{c}^{\,T}\vec{u} = \sum_{j=1}^{n} c_j u_j &\le \sum_{j=1}^{n} z_j u_j \\
&= \vec{z}^{\,T}\vec{u} \\
&= \vec{c}_m^{\,T}\,\underline{Y}\,\vec{u} && (\because \vec{z}^{\,T} = \vec{c}_m^{\,T}\underline{Y}) \\
&= \vec{c}_m^{\,T} B^{-1} A \vec{u} && (\because A = B\underline{Y}) \\
&= \vec{c}_m^{\,T} B^{-1} \vec{b} \\
&= \vec{c}_m^{\,T} \vec{x}_B \\
&= \text{current objective value}
\end{aligned}$$
