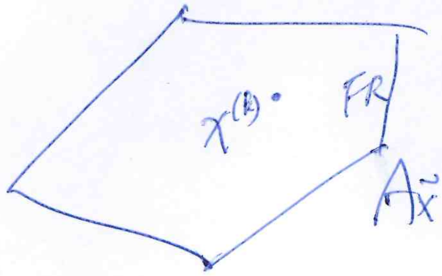
Karmarker

Interior pt method

$$\text{LPP}\quad \begin{cases} \max \vec{c}^T\vec{x} \\ A\vec{x}=\vec{b} \\ \vec{x}\geq\vec{0} \end{cases}$$

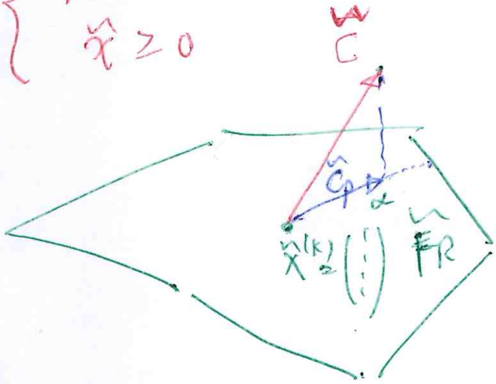


(i) $\vec{x}^{(k)}$ interior pt of FR

(ii) LPP $\rightarrow$ $\widetilde{\text{LPP}}$

$\vec{x}^{(k)} \rightarrow \tilde{x}^{(k)} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$$\widetilde{\text{LPP}}\quad \tilde{c}^T\tilde{x}$$
$$\begin{cases} \tilde{A}\tilde{x}=b \\ \tilde{x}\geq 0 \end{cases}$$



(iii) $\widetilde{\text{LPP}}$

$$\tilde{x}^{(k+1)} = \tilde{x}^{(k)} + \alpha\,\tilde{c}_p$$

(a) $\tilde{A}\tilde{x}^{(k+1)} = b$ $\quad \tilde{x}^{(k+1)} \in \widetilde{FR}$

(b) $\tilde{x}^{(k+1)} \in$ feasible & in the interior

$\wedge$ (all entries are positive)

Choose α

(iv) $\tilde{x}^{(k+1)} \rightarrow x^{(k+1)} \in FR.$

$\max \quad z = x_1 + 2x_2$ $\qquad \vec{c} = (1, 2)$

s.t. $\quad x_1 + x_2 + x_3 = 8$ $\qquad \begin{cases} A = (1\,1\,1) \quad \vec{b} = 8 \\ \vec{x} \geq \vec{0} \end{cases}$

$\quad x_1, x_2, x_3 \geq 0$

Iteration 1

(i) $\vec{x}^{(0)} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ is interior pt of FR.

$\vec{x} = D\hat{x}$

(ii) $\hat{x}_1 = \frac{1}{2} x_1$, $\quad \hat{x}_2 = \frac{1}{2} x_2 \quad \hat{x}_3 = \frac{1}{4} x_3$ $\qquad \hat{x} = D^{-1} x$

$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & & 0 \\ & \frac{1}{2} & \\ 0 & & \frac{1}{4} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\hat{x}^{(0)} = \begin{pmatrix} \hat{x}_1^{(0)} \\ \hat{x}_2^{(0)} \\ \hat{x}_3^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\max \quad z = 2\hat{x}_1 + 4\hat{x}_2 + 0\hat{x}_3$

s.t. $\begin{cases} 2\hat{x}_1 + 2\hat{x}_2 + 4\hat{x}_3 = 8 \\ \hat{x}_1, \hat{x}_2, \hat{x}_3 \geq 0 \end{cases}$ $\qquad \hat{c} = (2\,4\,0)$

$\hat{A}(2\,2\,4)$

(iii) $\hat{c}_p = (I - \hat{A}^t(\hat{A}\hat{A}^t)^{-1}\hat{A})\hat{c}$ $\qquad \hat{c}^T = (2, 4, 0) = \; \cdot (1, 2, 0) D$

$= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$\hat{x}^{(0)} \in FR \Rightarrow \hat{A}\hat{x}^{(0)} = b$

$\hat{x}^{(1)} = \hat{x}^{(0)} + \alpha\hat{c}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\alpha}{|-2|}\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ $\qquad 0 \leq \alpha \leq 1$

normalization

(a) $\hat{A}\hat{x}^{(1)} = b \quad \forall \alpha$

$\Leftrightarrow \hat{A}\hat{c}_p = (2\,2\,4)\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = 0$

(b) $\hat{x}^{(1)} \geq \vec{0}$

• (ii) $\alpha = 0.99$, in the lecture notes $\alpha = \frac{1}{2}$.

$$\breve{x}^{(1)} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}5/4\\7/4\\1/2\end{pmatrix} \in \breve{FR}$$

$$\Rightarrow \quad x^{(1)} = \begin{pmatrix}2\,\breve{x}_1^{(1)}\\2\,\breve{x}_2^{(1)}\\4\,\breve{x}_3^{(1)}\end{pmatrix} = \begin{pmatrix}5/2\\7/2\\2\end{pmatrix}$$

end of iteration 1,

$$\max\ \vec{c}^T\vec{x} \qquad \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0}\end{cases} \quad \xrightarrow[\vec{x} = D\breve{x}]{\breve{x} = D^{-1}\vec{x}} \quad \max\ \vec{c}^T D\breve{x} = \underbrace{(D\vec{c})^T}_{\breve{c}^T}\breve{x}$$

$$\begin{cases} \underbrace{AD}_{\breve{A}}\breve{x} = b \\ D\breve{x} \geq \vec{0}\end{cases} \Rightarrow \begin{cases} \breve{A}\breve{x} = b \\ \breve{x} \geq \vec{0}\end{cases}$$

$$\begin{cases} \breve{c}^T = \vec{c}^T D \\ \breve{A} = AD \end{cases}$$

Iteration 2

(i) $\vec{X}^{(1)} = \begin{pmatrix} 5/2 \\ 7/2 \\ 2 \end{pmatrix}$ interior pt of FR

(ii) $\vec{X}^{(1)} = D\,\tilde{X}^{(1)}$ $\quad \tilde{X}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} 5/2 & & \\ & 7/2 & \\ & & 2 \end{pmatrix} \qquad \tilde{x}_1 = \frac{2}{5}x_1, \; \tilde{x}_2 = \frac{2}{7}x_2 \; \tilde{x}_3 = \frac{1}{2}x_3$$

$$\tilde{A} = A \cdot D = (1\;1\;1)\begin{pmatrix} 5/2 & & \\ & 7/2 & \\ & & 2 \end{pmatrix} = (5/2, 7/2, 2)$$

$$\tilde{C}^T = \vec{C}^T \cdot D = (1\;2\;0)\begin{pmatrix} 5/2 & & \\ & 7/2 & \\ & & 2 \end{pmatrix} = (5/2, 7, 0)$$

(iii) $$\tilde{C}_p = (\underbrace{I - \tilde{A}^t(\tilde{A}\tilde{A}^t)^{-1}\tilde{A}}_{P})\tilde{C} = \begin{pmatrix} -11/12 \\ 133/60 \\ -41/15 \end{pmatrix} \qquad (\tilde{A}\tilde{C}_p = \vec{0})$$

$$\tilde{X}^{(2)} = \tilde{X}^{(1)} + \underbrace{\alpha}_{\text{const}}\,\tilde{C}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\alpha}{\left|\frac{41}{15}\right|}\begin{pmatrix} -11/12 \\ 133/60 \\ -41/15 \end{pmatrix} \qquad 0 \le \alpha \le 1$$

max in abs. value

$\alpha = 1/2$

$$\tilde{X}^{(2)} = \begin{pmatrix} 0.83 \\ 1.40 \\ 0.50 \end{pmatrix}$$

(a) $\tilde{A}\tilde{X}^{(2)} = 8$.

(b) $\alpha = \frac{1}{2}$ $\tilde{X}^{(2)} \ge \vec{0}$ interior

$\max \quad z = x_1 - 2x_2 \qquad \vec{c}^T = (1, -2, 0)$

$$\text{LPP}\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases} \qquad A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Iteration 1 initial solution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

To transform $\vec{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to $\tilde{x} = (1, 1, 1)$

$$\underset{\vec{x}}{\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}} = \underbrace{\begin{pmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & \frac{1}{3} \end{pmatrix}}_{D} \underset{\tilde{x}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$$\tilde{x} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} x$$

$\widetilde{\text{LPP}}$

$$\min \quad z = \tfrac{1}{3}\tilde{x}_1 - \tfrac{2}{3}\tilde{x}_2$$

$$\begin{cases} \frac{1}{3}\tilde{x}_1 - \frac{2}{3}\tilde{x}_2 + \frac{1}{3}\tilde{x}_3 = 0 \\ \frac{1}{3}\tilde{x}_1 + \frac{1}{3}\tilde{x}_2 + \frac{1}{3}\tilde{x}_3 = 1 \end{cases}$$

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \geq 0$

$$\tilde{A} = AD = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\tilde{c} = Dc = \begin{pmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix}$$

$$P = I - \tilde{A}^T (\tilde{A}\tilde{A}^T)^{-1} \tilde{A}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \left[\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \right]^{-1} \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\tilde{c}_p = P\tilde{c} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{pmatrix} \qquad \tilde{A}\tilde{c}_p = \vec{0}$$

$$\tilde{x}_{new} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\alpha}{(\frac{1}{6})} \begin{pmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(Note that $\tilde{A}\tilde{x}_{new} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ \forall \alpha$ & $\tilde{x}_{new} \geq 0 \ \forall \alpha \leq 1$)

$\begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}$

Let's choose $\alpha = \frac{1}{2}$, then new $\hat{x}$ is

$$\hat{x}_{new} = \begin{pmatrix}3/2\\1\\1/2\end{pmatrix} \Rightarrow \vec{x}_{new} = D\hat{x}_{new} = \begin{pmatrix}1/3&&\\&1/3&\\&&1/3\end{pmatrix}\begin{pmatrix}3/2\\1\\1/2\end{pmatrix} = \boxed{\begin{pmatrix}1/2\\1/3\\1/6\end{pmatrix}}$$

<u>Iteration 2:</u> Initial solution is $x = (1/2, 1/3, 1/6)$

To transform $\vec{x} = (1/2, 1/3, 1/6)$ to $\hat{x} = (1, 1, 1)$

$$\underset{\vec{x}}{\begin{pmatrix}1/2\\1/3\\1/6\end{pmatrix}} = \underset{D}{\begin{pmatrix}1/2&&\\&1/3&\\&&1/6\end{pmatrix}}\underset{\hat{x}}{\begin{pmatrix}1\\1\\1\end{pmatrix}} \Rightarrow \hat{x} = \begin{pmatrix}2&&\\&3&\\&&6\end{pmatrix}x$$

$$\Downarrow$$

$$\hat{A} = AD = \begin{pmatrix}1&-2&1\\1&1&1\end{pmatrix}\begin{pmatrix}1/2&&\\&1/3&\\&&1/6\end{pmatrix} = \begin{pmatrix}1/2&-2/3&1/6\\1/2&1/3&1/6\end{pmatrix}$$

$$\min \; Z = \tfrac{1}{2}\hat{x}_1 - \tfrac{2}{3}\hat{x}_2$$

$$\begin{cases}\tfrac{1}{2}\hat{x}_1 - \tfrac{2}{3}\hat{x}_2 + \tfrac{1}{6}\hat{x}_3 = 0\\ \tfrac{1}{2}\hat{x}_1 + \tfrac{1}{3}\hat{x}_2 + \tfrac{1}{6}\hat{x}_3 = 1\\ \hat{x}_1, \hat{x}_2, \hat{x}_3 \geq 0\end{cases}$$

$$\hat{c} = Dc = \begin{pmatrix}1/2&&\\&1/3&\\&&1/6\end{pmatrix}\begin{pmatrix}1\\-2\\0\end{pmatrix} = \begin{pmatrix}1/2\\-2/3\\0\end{pmatrix}$$

$$P = I - \hat{A}^T(\hat{A}\hat{A}^T)^{-1}\hat{A} = \begin{pmatrix}1/10&0&-3/10\\0&0&0\\-3/10&0&9/10\end{pmatrix}$$

$$\vec{c}_p = P\hat{c} = \begin{pmatrix}1/10&0&-3/10\\0&0&0\\-3/10&0&9/10\end{pmatrix}\begin{pmatrix}1/2\\-2/3\\0\end{pmatrix} = \begin{pmatrix}1/20\\0\\-3/20\end{pmatrix}$$

$$\hat{x}_{new} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{\alpha}{\left(\frac{3}{20}\right)}\begin{pmatrix}1/20\\0\\-3/20\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}1/3\\0\\-1\end{pmatrix}$$

(Note that $\hat{A}\hat{x}_{new} = \begin{pmatrix}0\\1\end{pmatrix} \;\forall \alpha$; $\hat{x}_{new} \geq 0 \;\forall \alpha \leq 1$)

Choosing $\alpha = \frac{1}{2}$, $\hat{x}_{new} = \begin{pmatrix}7/6\\1\\1/2\end{pmatrix} \Rightarrow \vec{x}_{new} = \begin{pmatrix}1/2&&\\&1/3&\\&&1/6\end{pmatrix}\begin{pmatrix}7/6\\1\\1/2\end{pmatrix} = \begin{pmatrix}7/12\\1/3\\1/12\end{pmatrix}$ #