Primal

$$\max \ \vec{c}^T\vec{x}$$

C.F. $\begin{cases} A\vec{x} \le \vec{b} \\ \vec{x} \ge \vec{0} \end{cases}$

Dual

$$\min \ \vec{b}^T\vec{u}$$

$\begin{cases} A^T\vec{u} \ge \vec{c} \\ \vec{u} \ge \vec{0} \end{cases}$

Thm 5.1 Dual of the dual is primal.

pf: Dual $\xrightarrow{CF}$ $\max \ -\vec{b}^T\vec{u}$

$$\begin{cases} -A^T\vec{u} \le -\vec{c} \\ \vec{u} \ge \vec{0} \end{cases} \quad CF$$

$\Big\downarrow$ definition

$$\min \ -\vec{c}^T\vec{x}$$

$$\text{st} \begin{cases} (-A^T)^T\vec{x} \ge -\vec{b} \\ \vec{x} \ge \vec{0} \end{cases}$$

$\Downarrow$

$$\max \ \vec{c}^T\vec{x}$$

$$\begin{cases} A\cdot\vec{x} \le \vec{b} \\ \vec{x} \ge \vec{0} \end{cases} \quad (= \text{primal})$$

#

$$\max \ \vec{c}^T\vec{x} \qquad \qquad \min \ \vec{b}^T\vec{u}$$

$$\text{s.t.} \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \ge \vec{0} \end{cases} \xrightarrow{\text{Dual}} \begin{cases} A^T\vec{u} \ge \vec{c} \\ \vec{u} \text{ is free} \end{cases}$$

$$\Downarrow$$

$$\max \ \vec{c}^T\vec{x}$$

$$\text{s.t.} \begin{cases} A\vec{x} \le \vec{b} \\ -A\vec{x} \le -\vec{b} \quad (\Leftarrow A\vec{x} \ge \vec{b}) \quad (\text{C.F.}) \\ \vec{x} \ge \vec{0} \end{cases}$$

$$\Downarrow \quad \max \ \vec{c}^T\vec{x}$$

$$\begin{cases} \begin{pmatrix} A \\ -A \end{pmatrix}\vec{x} \le \begin{pmatrix} \vec{b} \\ -\vec{b} \end{pmatrix} \\ \vec{x} \ge \vec{0} \end{cases}$$

$$\Big\downarrow \ \text{Dual}$$

$$\min \ (+b^T, -b^T)\begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix}$$

$$\begin{cases} (A^T, -A^T)\begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} \ge \vec{c} \\ \vec{u}_1, \vec{u}_2 \ge \vec{0} \end{cases}$$

$$\min \ \vec{b}^T(\vec{u}_1 - \vec{u}_2) \qquad \qquad \vec{u}_1 - \vec{u}_2 = \vec{u}$$

$$\Longleftrightarrow \begin{cases} A^T(\vec{u}_1 - \vec{u}_2) \ge \vec{c} \\ \vec{u}_1, \vec{u}_2 \ge \vec{0} \end{cases}$$

$$\min \ \vec{b}^T\vec{u}$$

$$\Longleftrightarrow \begin{cases} A^T\vec{u} \ge \vec{c} \\ \vec{u} \text{ is free} \end{cases}$$

P3

Example 5.3. Let us consider a primal given by

$$\begin{array}{ll} \min & 5x_1 + 6x_2 \\ \text{subject to} & \left\{ \begin{array}{l} x_1 + 2x_2 = 5 \\ -x_1 + 5x_2 \geq 3 \\ 4x_1 + 7x_2 \leq 8 \\ x_1 \text{ free, } x_2 \geq 0 \end{array} \right. \end{array}$$

$x_1 = x_1' - x_1''$ $x_1', x_1'' \geq 0$

The standardized primal is given by

$$\begin{array}{ll} \min & 5x_1' - 5x_1'' + 6x_2 + 0x_3 + 0x_4 \\ \text{subject to} & \left\{ \begin{array}{rrrrrl} x_1' - & x_1'' + & 2x_2 & & & = 5 \\ -x_1' + & x_1'' + & 5x_2 - & x_3 & & = 3 \\ 4x_1' - & 4x_1'' + & 7x_2 & & + x_4 & = 8 \\ x_1', & x_1'', & x_2, & x_3, & x_4 & \geq 0 \end{array} \right. \end{array}$$

S.F.

3×5

Since the primal is a minimization problem, the dual problem will be a maximization problem with $\leq$ signs. The dual variables are assumed to be free first.

Dual of SF

$$\begin{array}{ll} \max & 5u_1 + 3u_2 + 8u_3 \\ \text{subject to} & \left\{ \begin{array}{rrrl} u_1 - & u_2 + & 4u_3 & \leq 5 \\ -u_1 + & u_2 - & 4u_3 & \leq -5 \\ 2u_1 + & 5u_2 + & 7u_3 & \leq 6 \\ - & u_2 & & \leq 0 \\ & & u_3 & \leq 0 \\ u_1, & u_2, & u_3 & \text{free} \end{array} \right. \end{array}$$

negation

5×3

The first two inequality constraints combine together to give an equality constraint.

Simplification

$$\begin{array}{ll} \max & 5u_1 + 3u_2 + 8u_3 \\ \text{subject to} & \left\{ \begin{array}{rrrl} u_1 - & u_2 + & 4u_3 & = 5 \\ 2u_1 + & 5u_2 + & 7u_3 & \leq 6 \\ & u_2 & & \geq 0 \\ & & u_3 & \leq 0 \\ u_1, & u_2, & u_3 & \text{free} \end{array} \right. \end{array}$$

$u_2 \geq 0$

new $u_3 \geq 0$

By replacing the free variable u_1 by $u_1' - u_1''$ and the negative variable u_3 by $-u_3$, we finally arrive at

$$\begin{array}{ll} \max & 5u_1' - 5u_1'' + 3u_2 - 8u_3 \\ \text{subject to} & \left\{ \begin{array}{rrrrl} u_1' - & u_1'' - & u_2 - & 4u_3 & = 5 \\ 2u_1' - & 2u_1'' + & 5u_2 - & 7u_3 & \leq 6 \\ u_1', & u_1'', & u_2, & u_3 & \geq 0 \end{array} \right. \end{array}$$

which is the dual problem of the original primal problem.

Example 5.4. (Transportation Problem) Suppose that there are m sources that can provide materials to n destinations that require the materials. The following is called the *costs and requirements table* for the transportation problem.

$$
\begin{array}{c|cccc|c}
 & & \text{Destination} & & & \text{Supply} \\
\hline
 & c_{11} & c_{12} & \cdots & c_{1n} & s_1 \\
\text{Origin} & c_{21} & c_{22} & \cdots & c_{2n} & s_2 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & c_{m1} & c_{m2} & \cdots & c_{mn} & s_m \\
\hline
\text{Demand} & d_1 & d_2 & \cdots & d_n &
\end{array}
$$

where c_{ij} is the unit transportation cost from origin i to destination j, s_i is the supply available from origin i and d_j is the demand required for destination j. We assume that total supply equals to total demand, i.e.

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j .$$

The problem is to decide the amount x_{ij} to be shipped from i to j so as to minimize the total transportation cost while meeting all demands. That is

$$
\begin{array}{lll}
\min & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{subject to} & \left\{\begin{array}{l} \sum_{j=1}^{n} x_{ij} = s_i \\ \sum_{i=1}^{m} x_{ij} = d_j \\ x_{ij} \geq 0 \end{array}\right. & \begin{array}{l} (i = 1, 2, \cdots, m) \\ (j = 1, 2, \cdots, n) \\ (i = 1, 2, \cdots, m\ ; \quad j = 1, 2, \cdots, m) \end{array}
\end{array}
$$

The dual is then given by:

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j & \\
\text{subject to} & \left\{\begin{array}{ll} u_i + v_j & \leq c_{ij} \\ u_i\ , v_j & \text{free} \end{array}\right. & (i = 1, 2, \cdots, m\ ; \quad j = 1, 2, \cdots, n)
\end{array}
$$

5.2 Duality Theorems

We first give the relationship between the objective values of the primal and of the dual.

Theorem 5.2 (Weak Duality Theorem). *If* $\mathbf{x}$ *is a feasible solution (not necessarily basic) to the primal and* $\mathbf{u}$ *is a feasible solution (not necessarily basic) to the dual, then*

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u}\ .$$

Proof. Since $\mathbf{x}$ is a feasible solution to the primal P, we have $A\mathbf{x} \leq \mathbf{b}$. As $\mathbf{u} \geq \mathbf{0}$, we have

$$\mathbf{u}^T A\mathbf{x} \leq \mathbf{u}^T\mathbf{b} = \mathbf{b}^T\mathbf{u}. \tag{5.1}$$

Similarly, since $\mathbf{A}^T\mathbf{u} \geq \mathbf{c}$ and $\mathbf{x} \geq \mathbf{0}$, we have

$$\mathbf{x}^T A^T\mathbf{u} \geq \mathbf{x}^T\mathbf{c}.$$

Taking the transpose and combining with (5.1), we get $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u}$. □

$$\min \vec{c}^T \vec{x} \qquad \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0} \end{cases} \xrightarrow{\text{Dual}} \max \vec{b}^T \vec{u} \qquad \begin{cases} A^T\vec{u} \leq \vec{b} \\ \vec{u} \text{ is free} \end{cases}$$

Eg 5.3

$$\min 5x_1 + 6x_2 \qquad \begin{cases} x_1 + 2x_2 = 5 \\ -x_1 + 5x_2 \geq 3 \\ 4x_1 + 7x_2 \leq 8 \\ x_1 \text{ free} \\ x_2 \geq 0 \end{cases}$$

$$\max \quad 5u_1 + 3u_2 + 8u_3$$

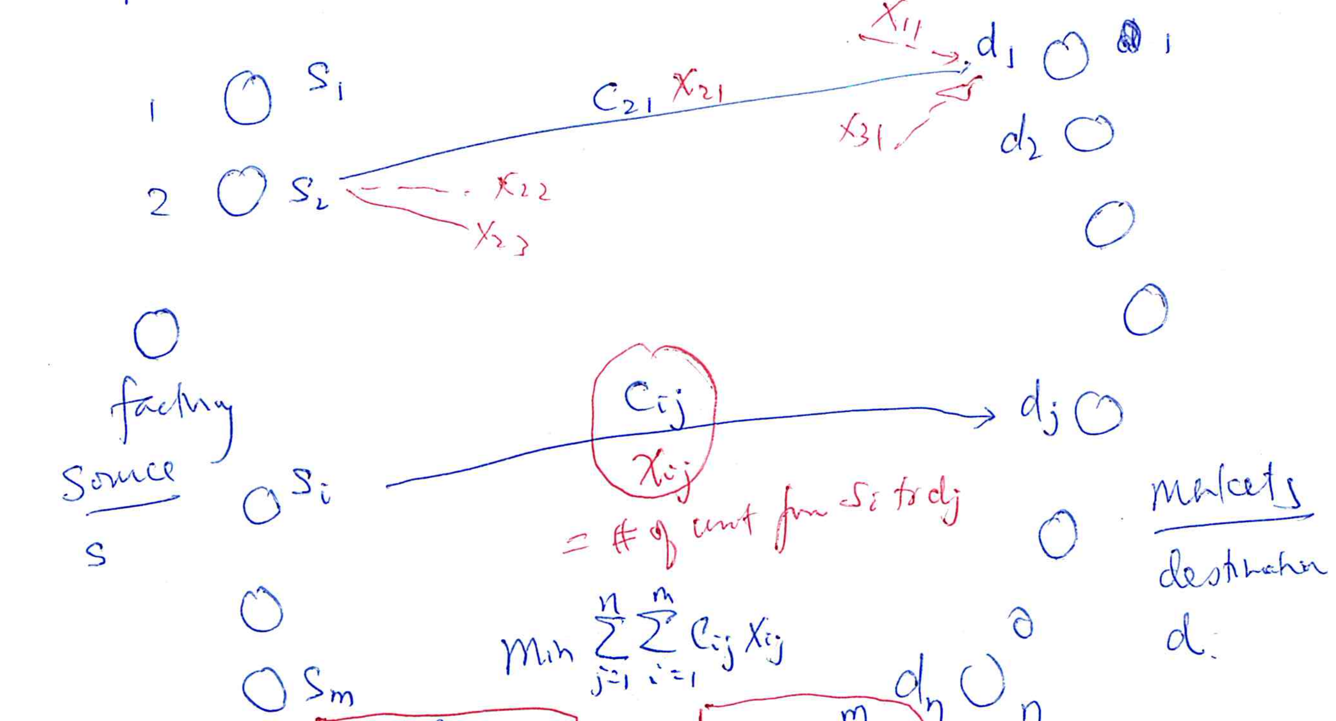
$$\begin{cases} 1u_1 - 1u_2 + 4u_3 = 5 \\ 2u_1 + 5u_2 + 7u_3 \leq 6 \\ u_1 \text{ free}, \ u_2 \geq 0, \ u_3 \leq 0 \end{cases}$$

$u_1 = u_1' - u_1''$, $u_1', u_1'' \geq 0$ $\quad u_3 = -\hat{u}_3$

$\max \ 5u_1' - 5u_1'' + 3u_2 - 8\hat{u}_3$ $\quad \hat{u}_3 \geq 0$

$$\begin{cases} u_1' - u_1'' - u_2 - 4\hat{u}_3 = 5 \\ 2(u_1' - u_1'') + 5u_2 - 7\hat{u}_3 \leq 6 \\ u_1', u_1'', u_2, \hat{u}_3 \geq 0 \end{cases}$$

Transportation Problem



x_{ij} = # of unit from S_i to d_j

$$\min \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$$

$$\boxed{S_i = \sum_{j=1}^{n} x_{ij}} \qquad \boxed{d_j = \sum_{i=1}^{m} x_{ij}}$$

$$\text{Min} \sum_i \sum_j c_{ij} x_{ij}$$

s.t.

$$\begin{cases} \sum_i x_{ij} = d_j & \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} \\ \sum_j x_{ij} = s_i & \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} \begin{matrix} u_1 \\ \vdots \\ u_m \end{matrix} \\ x_{ij} \ge 0 \end{cases}$$

$$\max \sum_{i=1}^{m} s_i u_i + \sum_{i=1}^{n} d_i v_i$$

s.t.

$$A^T \begin{pmatrix} u_1 \\ \vdots \\ u_m \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \le \begin{pmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \\ c_{21} \\ \vdots \\ c_{mn} \end{pmatrix}$$

$$\begin{cases} u_i \text{ free} \\ v_i \text{ free} \end{cases}$$

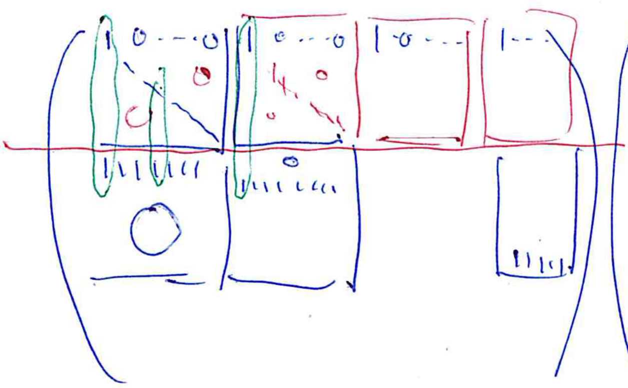
$m+n$ variables

$m \times n$ variables

10 10

$m+n$ constraints

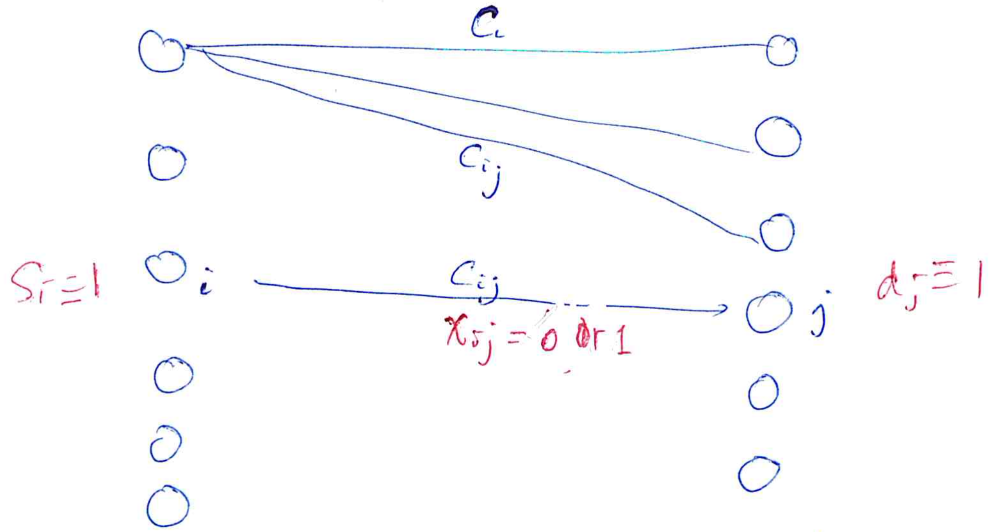
20 constraints



$$\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \\ \vdots \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_m \\ d_1 \\ \vdots \\ d_n \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & \vdots & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \\ v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \vdots \\ u_i + v_j \\ \vdots \end{pmatrix} \le \begin{pmatrix} c_{ij} \end{pmatrix}$$

Assignment Problem

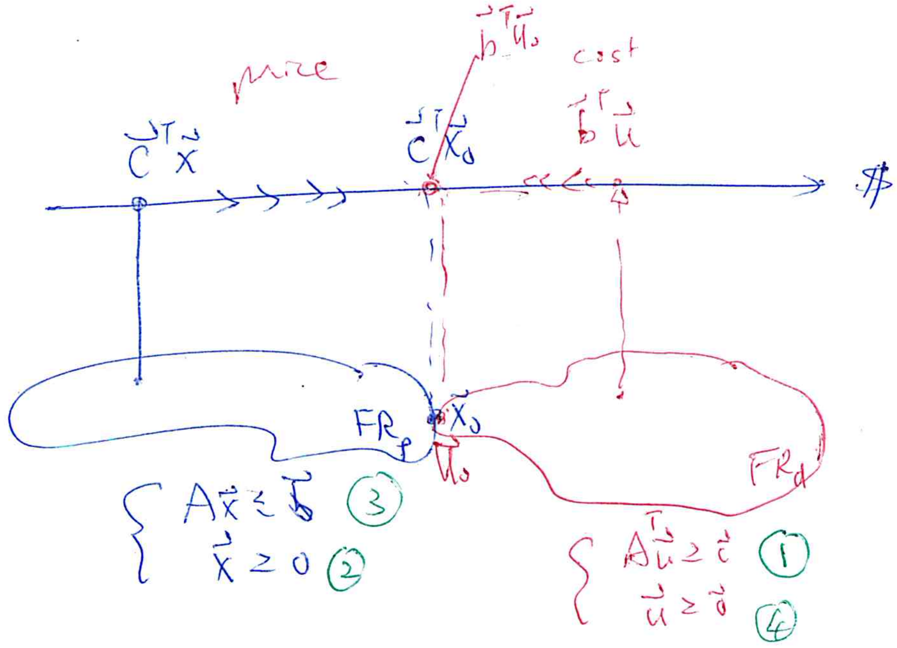


m men m tasks

repairmen escalators

Thm 5.2

Weak Duality Thm



If $\vec{x} \in FR_p$, $\vec{u} \in FR_d$

$$\vec{c}^T\vec{x} \le \vec{b}^T\vec{u}$$

Pf: $\vec{c}^T\vec{x} = \vec{x}^T\vec{c}$ $\quad (\sum x_i c_i)$

$\overset{①②}{\le} \vec{x}^T A^T \vec{u}$ $\quad (\because \vec{x} \ge \vec{0}\ \ A^T\vec{u} \ge \vec{c})$

$= \vec{u}^T (A^T)^T \vec{x}$

$= \vec{u}^T A \vec{x}$

$\overset{③④}{\le} \vec{u}^T \vec{b}$ $\quad (\because \vec{u} \ge \vec{0}\ \ A\vec{x} \le \vec{b})$

$= \vec{b}^T\vec{u}$ #

Coro Thm 5.3 If $\vec{x}_0 \in FR_p$ $\vec{u}_0 \in FR_d$ & $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$ then $\vec{x}_0$ & $\vec{u}_0$ are optimal solution

Pf: $\forall \vec{x} \in FR_p$ $\quad \vec{c}^T\vec{x} \overset{\text{Thm 5.2}}{\le} \vec{b}^T\vec{u}_0$ $\quad (\because \vec{u}_0 \in FR_d)$

$$\vec{b}^T\vec{u}_0 = \vec{c}^T\vec{x}_0 \Rightarrow \vec{x}_0 \text{ is optimal.} \#$$

Thm 5.4

Strong Duality Th.

$$\{\vec{x}_0 \in FR_p,\ \vec{u}_0 \in FR_d \ \&\ \vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0\}$$

$\xRightarrow{\text{Thm 5.3}}$

$\Longleftrightarrow$ $\vec{x}_0$ is optimal & $\vec{u}_0$ is optimal

$\xLeftarrow[\text{Thm 5.4}]{}$

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pf. Primal / Dual

$$\max\ \vec{c}^T\vec{x} = \vec{c}^T\vec{x} + \overset{0}{\overset{\parallel}{\vec{c}_s}}\vec{x}_s \qquad \min\ \vec{b}^T\vec{u}$$

$$\begin{cases} A\vec{x} \le \vec{b} \iff [A \mid I]\begin{bmatrix}\vec{x}\\ \vec{x}_s\end{bmatrix} = \vec{b} \\ \vec{x} \ge \vec{0}\end{cases} \qquad \begin{cases} A^T\vec{u} \ge \vec{c} \\ \vec{u} \ge \vec{0}\end{cases}$$

$\vec{x}_s$: slack

$\vec{x}_0$ is optimal in LPP primal $\Rightarrow$ $\boxed{\vec{z} - \begin{pmatrix}\vec{c}\\ \vec{c}_s\end{pmatrix} \ge \vec{0}}$ (i), $\vec{c}_s = 0$

optimal

$$\begin{array}{c|c|c} x_{B_1} & & \\ \vdots & Y & B^{-1}\vec{b} = \vec{x}_B \\ x_{B_m} & & \\ \hline & z_j - c_j \ge 0 & \vec{c}^T\vec{x} = \vec{c}_B^T\vec{x}_B \end{array}$$

$\vec{z}^T = \vec{c}_B^T Y$ (ii)

$BY = [A \mid I]$ (iii)

$\vec{b} = [A \mid I]\begin{bmatrix}\vec{x}\\ \vec{x}_s\end{bmatrix}$

some columns inside $[A \mid I]$

$= [B \vdots R]\begin{bmatrix}\vec{x}_B\\ \vec{0}\end{bmatrix}$

$= B\vec{x}_B$

original tableau

$$\begin{array}{c|c:c|c} x_{s_1} & & & \\ x_{s_2} & A & I & \\ \vdots & & & \\ x_{s_m} & & & \\ \hline & -\vec{c} & -\vec{c}_s = 0 & \end{array}$$

$(\vec{c}, \vec{c}_s)$ in $(\vec{c}_B \mid \vec{c}_R)$

optimal value $= \vec{c}^T\vec{x}_0$

$= (\vec{c}_B^T, \vec{c}_R^T)\begin{pmatrix}\vec{x}_B\\ \vec{x}_R\end{pmatrix} = \vec{c}_B^T\vec{x}_B$