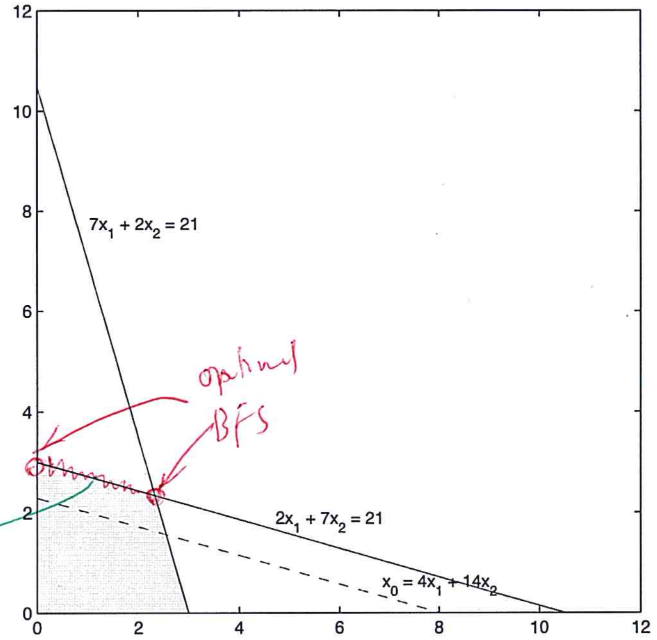
p1



optimal FS but not basic

Figure 4.4. *Infinitely many Optimal Solutions*

Any $(x_1, x_2) = (1, k)$ for k being any positive number is a feasible solution.

Example 4.4. Consider

$$\begin{array}{ll} \max & x_0 = 4x_1 + 14x_2 \\ \text{subject to} & \begin{cases} 2x_1 + 7x_2 \le 21 \\ 7x_1 + 2x_2 \le 21 \\ x_1, x_2 \ge 0 \end{cases} \end{array}$$

	x_1	x_2	x_3	x_4	**b**
x_3	2	7*	1	0	21
x_4	7	2	0	1	21
x_0	−4	−14	0	0	0

↓

nonbasic

	x_1	x_2	x_3	x_4	**b**
x_2	2/7	1	1/7	0	3
x_4	45/7*	0	−2/7	1	15
x_0	$\boxed{0}$	0	2	0	42

Optimal ≥ 0

0.0000001

optimal

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 15 \end{pmatrix}$$

$X_0 = 42$

P2

$\downarrow\uparrow$

	x_1	x_2	x_3	x_4	**b**
x_2	0	1	7/45	−2/45	7/3
x_1	1	0	−2/45	7/45*	7/3
x_0	0	0	2	$\boxed{0}$	42

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 7/3 \\ 0 \\ 0 \end{pmatrix}$

$x_0 = 42$

Thus all convex combinations of the points $[0, 3, 0, 15]$ and $[7/3, 7/3, 0, 0]$ are optimal feasible solutions.

4.4 Degeneracy and Cycling

Degenerate basic solutions are basic solutions with one or more basic variables at zero level. Degeneracy occurs when one or more of the constraints are redundant.

Example 4.5. Consider the following LLP

$$\max \quad x_0 = 2x_1 + x_2$$

$$\text{subject to} \begin{cases} 4x_1 + 3x_2 \le 12 \\ 4x_1 + x_2 \le 8 \\ 4x_1 - x_2 \le 8 \\ x_1,\ x_2 \ge 0 \end{cases}$$

$4x_1 + 3x_2 + x_3 = 12$

$4x_1 + x_2 + x_4 = 8$

$4x_1 - x_2 + x_5 = 8$

3

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	4	3	1	0	0	12
x_4	4**	1	0	1	0	8
x_5	4*	−1	0	0	1	8
x_0	−2	−1	0	0	0	0

5

3 nonzero basic variables

2 zero non-basic variables

(*) $\swarrow$ $\searrow$ (**)

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	4	1	0	−1	4
x_4	0	2*	0	1	−1	0
x_1	1	−1/4	0	0	1/4	2
x_0	0	−3/2	0	0	1/2	4

Degenerate Vertex { $x_4 = 0$ and basic }

$\downarrow$

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	2**	1	−1	0	4
x_1	1	1/4	0	1/4	0	2
x_5	0	−2	0	−1	1	0
x_0	0	−1/2	0	1/2	0	4

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \\ 0 \\ 0 \end{pmatrix}$

Degenerate Vertex { $x_5 = 0$ and basic }

$\downarrow$

P3

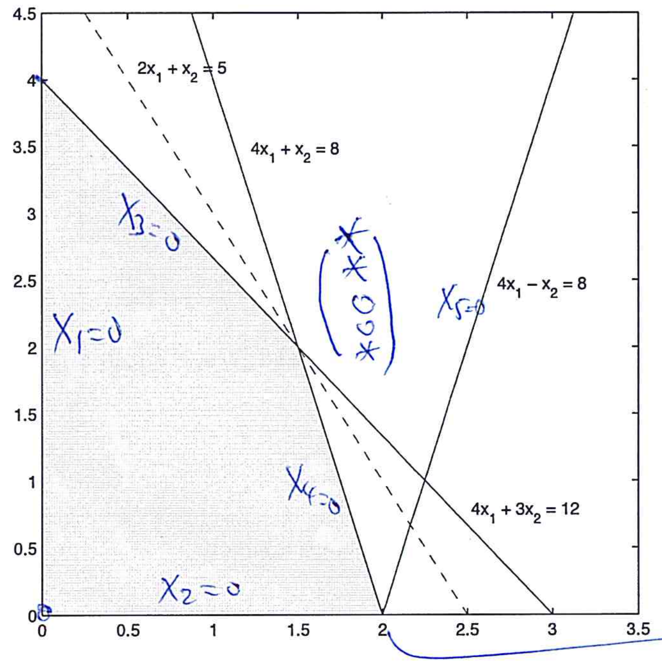


Figure 4.5. *Degeneracy and Cycling*

	x_1	x_2	x_3	x_4	x_5	b
x_3	0	0	1	-2	1^*	4
x_2	0	1	0	$1/2$	$-1/2$	0
x_1	1	0	0	$1/8$	$1/8$	2
x_0	0	0	0	$3/4$	$-1/4$	4

	x_1	x_2	x_3	x_4	x_5	b
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_5	0	0	1	-2	1	4
x_0	0	0	$1/4$	$1/4$	0	5

Degenerate Vertex { $x_2 = 0$ and basic }

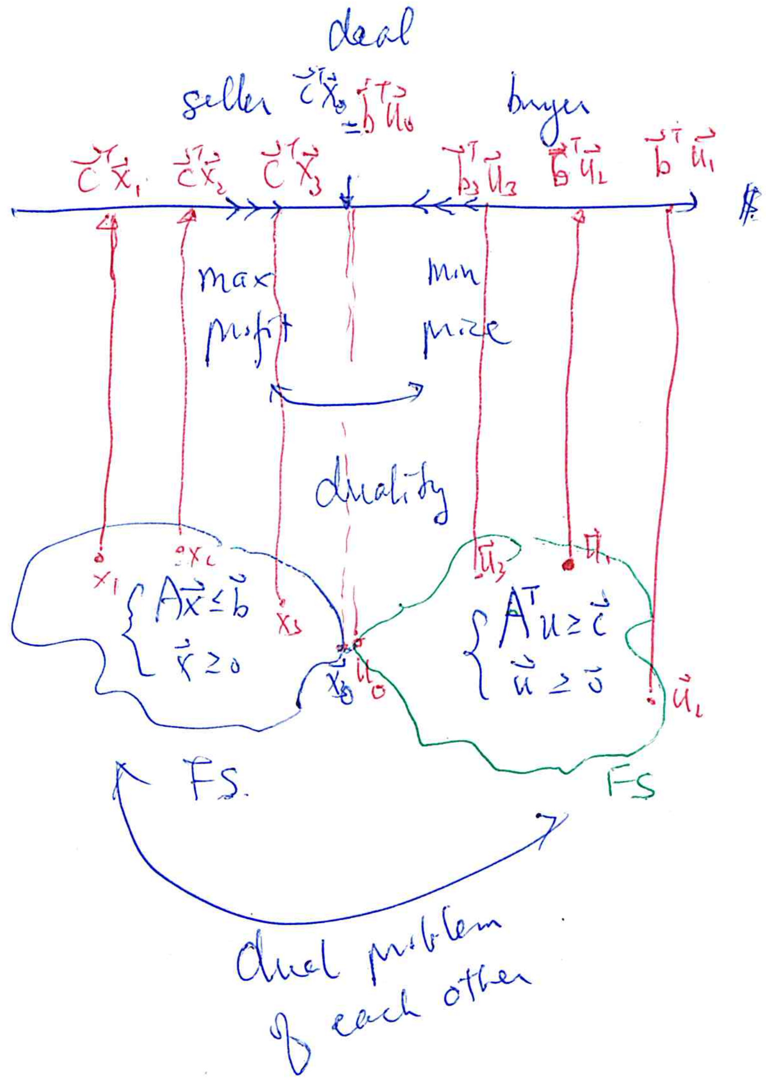
↓

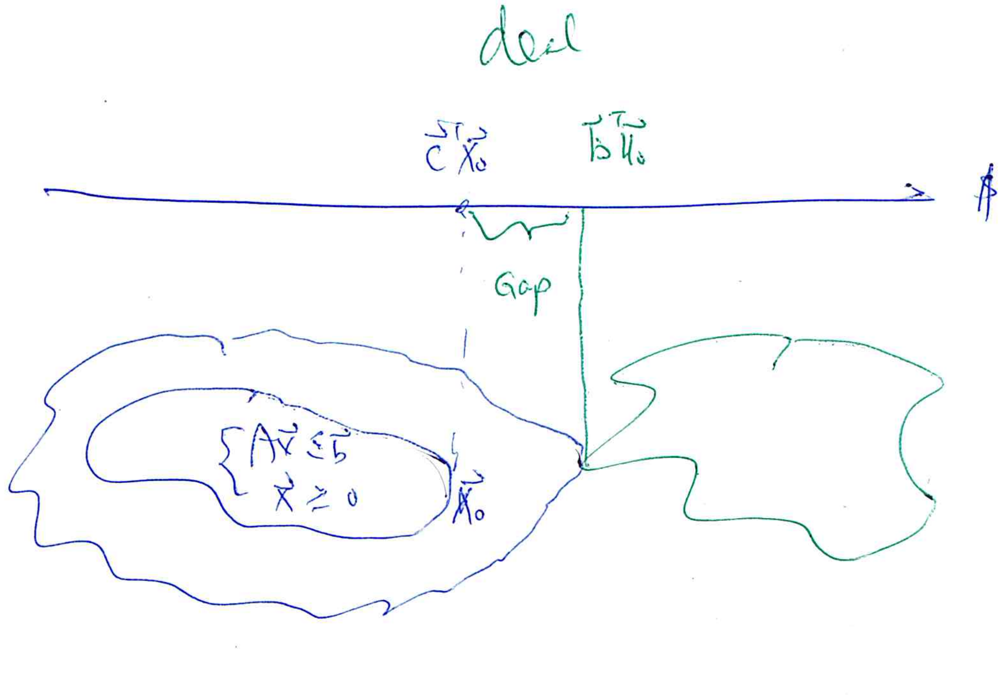
	x_1	x_2	x_3	x_4	x_5	b
x_5	0	0	1	-2	1	4
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_0	0	0	$1/4$	$1/4$	0	5

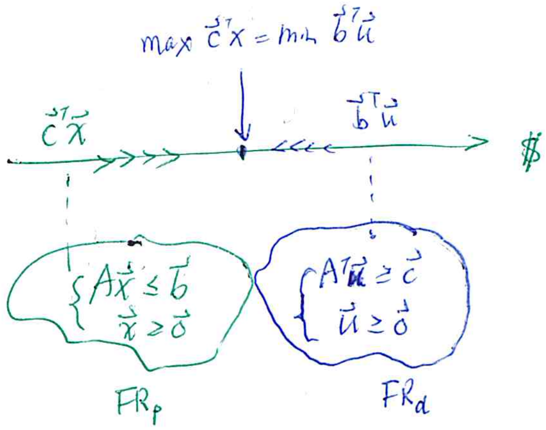
In figure 4.5, we see that the degenerate vertex V can be represented either by

$$\{x_2 = 0, x_4 = 0\}, \quad \{x_4 = 0, x_5 = 0\} \quad \text{or} \quad \{x_2 = 0, x_5 = 0\}.$$

We note that degeneracy guarantees the existence of more than one feasible pivot element, i.e. *tie-ratios* exist. For example, in the first tableau, the ratios for variables x_4 and x_5 are both equal to 2.


deal
seller $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$ buyer
$\vec{c}^T\vec{x}_1$ $\vec{c}^T\vec{x}_2$ $\vec{c}^T\vec{x}_3$ $\vec{b}^T\vec{u}_3$ $\vec{b}^T\vec{u}_2$ $\vec{b}^T\vec{u}_1$
\$
max profit
min price
duality
x_1 x_2 x_3 x_0
$\begin{cases} A\vec{x} \le \vec{b} \\ \vec{x} \ge 0 \end{cases}$
$\vec{u}_3$ $\vec{u}_1$ $\vec{u}_2$ $\vec{u}_0$
$\begin{cases} A^T u \ge \vec{c} \\ \vec{u} \ge \vec{0} \end{cases}$
FS.
FS
dual problem
of each other


deal
$\vec{c}^T\vec{x}_0$ $\vec{b}^T\vec{u}_0$
\$
Gap
$\begin{cases} A\vec{x} \le \vec{b} \\ \vec{x} \ge 0 \end{cases}$
$\vec{x}_0$



Chapter 5

DUALITY

5.1 The Dual Problems

Steve Smale 80 pills

Every linear programming problem has associated with it another linear programming problem and that the two problems have such a close relationship that whenever one problem is solved, the other is solved as well. The original LPP is called the *primal* problem and the associated LPP is called the *dual* problem. Together they are called a "*dual pair*" (primal + dual) in the sense that the dual of the dual will again be the primal.

Example 5.1. (The Diet Problem) How can a dietician design the most economical diet that satisfies the basic daily nutritional requirements for a good health? For simplicity, we assume that there are only two foods F_1 and F_2 and the daily nutrition required are N_1, N_2 and N_3. The unit cost of the foods and their nutrition values together with the daily requirement of each nutrition are given in the following table.

Veg. Meat

\$$u_1$ \$$u_2$ \$$u_3$

P V C

	x_1 F_1	F_2 x_2	Daily Requirement
$\boxed{u_1 + 2u_2 + 3u_3} \le 120$ Cost	120	180	–
Protein N_1	1	1	10
Vitamin N_2	2	4	24
Carbohydrate N_3	3	6	32

1p 1V 1C

10 24 32

Let x_j, $j = 1, 2$ be the number of units of F_j that one should eat in order to minimize the cost and yet fulfill the daily nutrition requirement. Thus the problem is to select the x_j such that

$$\min \qquad x_0 = 120x_1 + 180x_2$$

subject to the nutritional constraints:

$$\begin{cases} x_1 + \ \ x_2 \ge 10 \\ 2x_1 + 4x_2 \ge 24 \\ 3x_1 + 6x_2 \ge 32 \end{cases}$$

and the non-negativity constraints: $x_j \ge 0$, $j = 1, 2$. In matrix form, we have

$$\min \qquad x_0 = \mathbf{c}^T\mathbf{x} \qquad \overset{c}{(120, 180)}$$

$$\text{subject to} \begin{cases} A\mathbf{x} \ge \mathbf{b} \\ \mathbf{x} \ge \mathbf{0} \end{cases} \qquad \underset{A}{\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ge \underset{b}{\begin{pmatrix} 10 \\ 24 \\ 32 \end{pmatrix}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

P6

where

$$\mathbf{c} = \begin{bmatrix} 120 \\ 180 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 10 \\ 24 \\ 32 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

Now let us look at the same problem from a pharmaceutical company's point of view. How can a pharmaceutical company determine the price for each unit of nutrient pill so as to maximize revenue, if a synthetic diet made up of nutrient pills of various pure nutrients is adopted? Thus we have three types of nutrient pills P_1, P_2 and P_3. We assume that each unit of P_i contains one unit of the N_i. Let u_i be the unit price of P_i, the problem is to maximize the total revenue u_0 from selling such a synthetic diet, i.e.

$$\max \qquad u_0 = 10u_1 + 24u_2 + 32u_3$$

subject to the constraints that the cost of a unit of synthetic food j made up of P_i is no greater than the unit market price of F_j:

$$\begin{cases} u_1 + 2u_2 + 3u_3 \le 120 \\ u_1 + 4u_2 + 6u_3 \le 180 \\ u_1, u_2, u_3 \ge 0 \end{cases} \qquad \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 6 \end{pmatrix}}_{A^T} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \le \begin{pmatrix} 120 \\ 180 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \ge \vec{0}$$

In matrix form, the problem is:

$$\max \qquad u_0 = \mathbf{b}^T\mathbf{u} \qquad (10, 24, 32)$$

$$\text{subject to} \begin{cases} A^T\mathbf{u} \le \mathbf{c} \rightarrow \begin{pmatrix} 120 \\ 180 \end{pmatrix} \\ \mathbf{u} \ge \mathbf{0} \end{cases}$$

We said the two problems form a dual pair of linear programming problem, and we will see that the solution to one should lead to the solution of the other.

Definition 5.1. Let $\mathbf{x}$ and $\mathbf{c}$ be column n-vectors, $\mathbf{b}$ and $\mathbf{u}$ be column m-vectors and A be an m-by-n matrix. The *primal* and the *dual* problems can be defined as follows:

Primal CF	Dual
max $\mathbf{c}^T\mathbf{x}$	min $\mathbf{b}^T\mathbf{u}$
subject to $A\mathbf{x} \le \mathbf{b}$	subject to $A^T\mathbf{u} \ge \mathbf{c}$
$\mathbf{x} \ge \mathbf{0}$	$\mathbf{u} \ge \mathbf{0}$

	Primal	dual
(i)	cost coeff $\vec{c}$	r.h.s.
(ii)	max	min
(iii)	r.h.s. $\vec{b}$	cost coeff
(iv)	$\le$	$\ge$

Calling one primal and the other one dual is completely arbitrary for we have the following theorem.

Theorem 5.1. *The dual of the dual is the primal.*

Proof. Transforming the dual into canonical form, we have

$$\max \qquad u_0' = -\mathbf{b}^T\mathbf{u} \qquad \text{FC} \xrightarrow{\text{Dual}}$$

$$\text{subject to} \begin{cases} -A^T\mathbf{u} \le -\mathbf{c} \\ \mathbf{u} \ge \mathbf{0} \end{cases}$$

$$\boxed{\left((\text{primal})^{\text{dual}}\right)^{\text{dual}} = \text{primal}}$$