Chapter 4

SPECIAL CASES IN APPLYING SIMPLEX METHODS

4.1 No Feasible Solutions

In terms of the methods of artificial variable techniques, the solution at optimality could include one or more artificial variables at a positive level (i.e. as a non-zero basic variable). In such a case the corresponding constraint is violated and the artificial variable cannot be driven out of the basis. The feasible region is thus empty.

Example 4.1. Consider the following linear programming problem.

$$\boxed{\begin{array}{ll} \text{max} & x_0 = 2x_1 + x_2 \\ \text{subject to} & \left\{ \begin{array}{r} -x_1 + x_2 \geq 2 \\ x_1 + x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{array} \right. \end{array}} + 0X_3 - MX_4 + 0X_5$$

Using a surplus variable x_3, an artificial variable x_4 and a slack variable x_5, the augmented system is:

surplus artificial

$$\begin{array}{rcl} -\ x_1 + x_2 - x_3 + x_4 & = 2 \\ x_1 + x_2 \qquad\qquad + x_5 & = 1 \\ x_0 - 2x_1 - x_2 \qquad + Mx_4 \qquad & = 0 \end{array}$$

slack

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

Now the columns corresponding to x_4 and x_5 form an identity matrix. In tableau form, we have

	x_1	x_2	x_3	x_4	x_5	b
x_4	-1	1	-1	1	0	2
x_5	1	1	0	0	1	1
x_0	-2	-1	0	M	0	0

After elimination of the M in the x_4 column, we have the initial tableau:

starting simplex BFS tableau

	x_1	x_2	x_3	x_4	x_5	b	
x_4	-1	1	-1	1	0	2	2/1
← x_5	1	1*	0	0	1	1	1/1
x_0	$-2+M$	$-1-M$	M	0	0	$-2M$	

$X_1, X_3 = 0$ ↑ nonbasic now

X_2 nonbasic now $X_2 = 0$

X_2 basic next time X_2 ↑

$X_1, X_5 = 0$ nonbasic next time $\Rightarrow \begin{cases} X_2 \leq 2 \\ X_2 \leq 1 \end{cases} \Rightarrow \underline{X_2 \leq 1}$

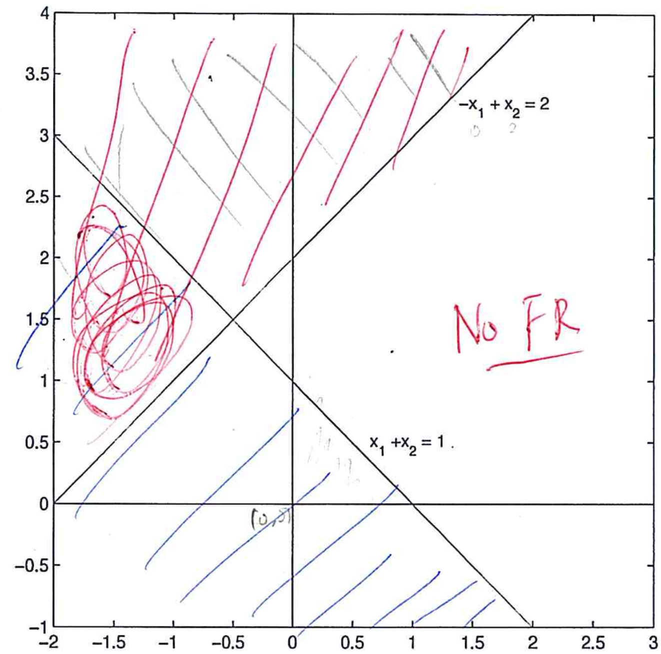


Figure 4.1. *No Feasible Region*

slack

	x_1	x_2	x_3	x_4	x_5	b
x_4	-2	0	-1	1	-1	1
x_2	1	1	0	0	1	1
x_0	$-1+2M$	0	M	0	$1+M$	$1-M$

≥ 0

optimal tableaux

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is still not BFS of the original problem

Since M is a very large number, $-1+2M$ is positive. Hence all entries in the x_0 row are nonnegative. Thus we have reached an optimal point. However, we see that the artificial variable $x_4 = 1$, which is not zero. That means that the solution just found is not a solution to our original problem. Indeed the $\mathbf{x}$ that satisfies $A\mathbf{x} + I\mathbf{x}_a = \mathbf{b}$ with $\mathbf{x}_a \neq 0$ is not a solution to $A\mathbf{x} = \mathbf{b}$. Figure 4.1 shows that the feasible region to the problem is empty.

4.2 Unbounded Solutions

Theorem 4.1. *Consider an LPP in feasible canonical form. If in the simplex tableau, there exists a nonbasic variable x_j such that $y_{ij} \leq 0$ for all $i = 1, 2, \cdots, m$, i.e. all entries in the x_j column are non positive, then the feasible region is unbounded. If moreover that $z_j - c_j < 0$, then there exists a feasible solution with at most $m+1$ variables nonzero and the corresponding value of the objective function can be set arbitrarily large.*

Proof. Let $\mathbf{x}_B$ be the current BFS with $B\mathbf{x}_B = \mathbf{b}$. Let the columns of B be denoted by $\mathbf{b}_i$. Then we have

$$B\mathbf{x}_B = \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i = \mathbf{b}.$$

Let $\mathbf{a}_j$ be the column of A that corresponds to the variable x_j. By (??), we have

$$\mathbf{a}_j = B\mathbf{y}_j = \sum_{i=1}^{m} y_{ij}\mathbf{b}_i.$$

Hence for all $\theta > 0$, we have

$$\begin{aligned}
\mathbf{b} &= \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i - \theta\mathbf{a}_j + \theta\mathbf{a}_j \\
&= \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i - \theta\sum_{i=1}^{m} y_{ij}\mathbf{b}_i + \theta\mathbf{a}_j \\
&= \sum_{i=1}^{m}(x_{B_i} - \theta y_{ij})\mathbf{b}_i + \theta\mathbf{a}_j\ .
\end{aligned}$$

Thus we obtain a new nonbasic solution of $m+1$ nonzero variables. This solution is feasible as

$$x_{B_i} - \theta y_{ij} \geq 0, \quad \text{for all } i.$$

Moreover, the value of x_j, which is equal to θ, can be set arbitrarily large, indicating that the feasible region is unbounded in the x_j direction.

If moreover that $c_j > z_j$, then the value of the objective function can be set arbitrarily large since

$$\begin{aligned}
\hat{z} &= \sum_{i=1}^{m}(x_{B_i} - \theta y_{ij})c_{B_i} + \theta c_j = \sum_{i=1}^{m} x_{B_i}c_{B_i} - \theta\sum_{i=1}^{m} y_{ij}c_{B_i} + \theta c_j \\
&= \mathbf{c}_B\mathbf{x}_B - \theta\mathbf{c}_B^T\mathbf{y}_j + \theta c_j = z - \theta z_j + \theta c_j = z + \theta(c_j - z_j).
\end{aligned}$$

This proves our assertion. □

Example 4.2. This is an example where the feasible region and the optimal value of the objective function are unbounded. Consider the LPP

$$\begin{array}{ll}
\max & x_0 = 2x_1 + x_2 \\
\text{subject to} & \begin{cases} x_1 - x_2 \leq 10 \\ 2x_1 - x_2 \leq 40 \\ x_1, x_2 \geq 0 \end{cases}
\end{array}$$

$\nearrow \infty$ $\quad x_2 \nearrow \infty$

FCF

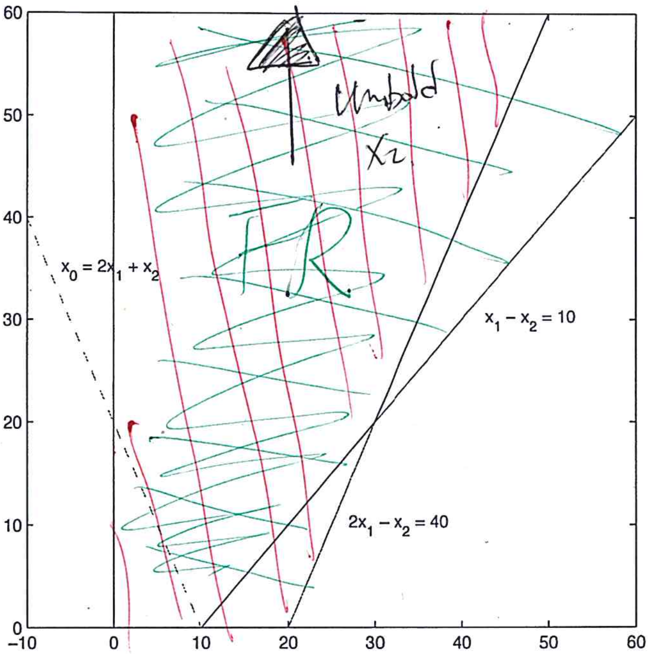
The initial tableau is

	x_1	x_2	x_3	x_4	b
x_3	1	-1	1	0	10
x_4	2	-1	0	1	40
x_0	-2	-1	0	0	0

NO ratio

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 40 \end{pmatrix} \downarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \boxed{>0} \end{pmatrix}$$

No positive ratio exists in x_2 column. Hence x_2 can be increased without bound while maintaining feasibility. It is evident from Figure 4.2.



FR unbdd
X_0 unbdd

Figure 4.2. *Unbound Feasible Region with Bounded Optimal Value*

Example 4.3. The following is an example where the feasible region is unbounded yet the optimal value is bounded. Consider the LPP

$$\max \quad x_0 = 6x_1 - 2x_2$$

$$\text{subject to} \begin{cases} 2x_1 - x_2 \le 2 \\ x_1 \le 4 \\ x_1, \quad x_2 \ge 0 \end{cases}$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \end{pmatrix}$$

The computation goes as follows

Unbdded in X_2

	x_1	x_2	x_3	x_4	b
x_3	2*	−1	1	0	2
x_4	1	0	0	1	4
x_0	−6	2	0	0	0

$\frac{2}{2} = 1$, $\frac{4}{1} = 4$

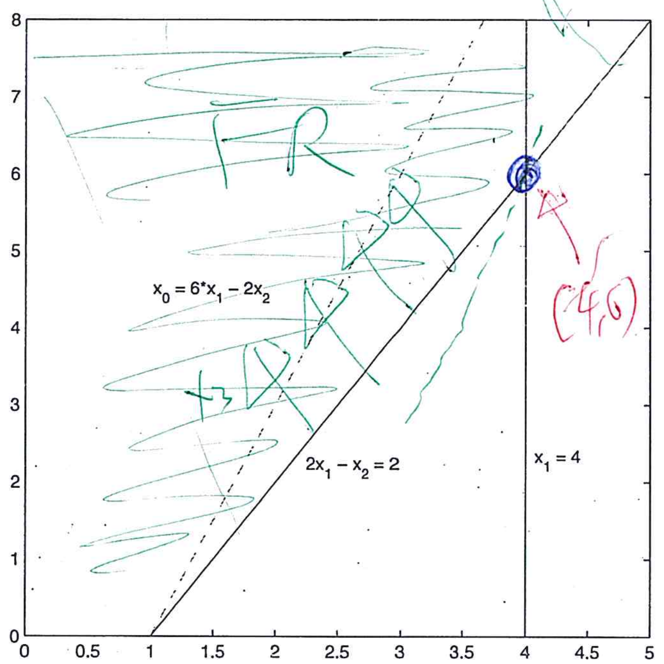
$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

$\begin{cases} X_1 \le 1 \\ X_1 \le 4 \end{cases} \Rightarrow X_1 \le 1$

↓

	x_1	x_2	x_3	x_4	b
x_1	1	−1/2	1/2	0	1
x_4	0	1/2*	−1/2	1	3
x_0	0	−1	3	0	6

$\frac{3}{1/2} =$



FR
unbdd
x_0 bdd

Figure 4.3. *Unbounded Feasible Region with Unbounded Optimal Value*
Any $(x_1, x_2) = (1, k)$ for k being any positive number is a feasible solution.

FR unbdd in x_3

unbdd in x_3

finite optimum

	x_1	x_2	x_3	x_4	b
x_1	1	0	0	1	4
x_2	0	1	−1	2	6
x_0	0	0	2	2	12

Optimal tableau

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \\ 0 \end{pmatrix}$$

4.3 Infinite Number of Optimal Solutions

Zero reduced cost coefficients for non-basic variables at optimality indicate alternative optimal solutions, since if we pivot in those columns, x_0 value remains the same after a change of basis for a different BFS, see Section 3.5. Notice that simplex method yields only the extreme point optimal (BFS) solutions. More generally, the set of alternative optimal solutions is given by the convex combination of optimal extreme point solutions. Suppose $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^p$ are extreme point optimal solutions, then $\mathbf{x} = \sum_{k=1}^{p} \lambda_k \mathbf{x}^k$, where $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{p} \lambda_k = 1$ is also an optimal solution. In fact, if $\mathbf{c}^T\mathbf{x}^k = z_0$ for $1 \le k \le p$, then

$$\mathbf{c}^T\mathbf{x} = \sum_{k=1}^{p} \lambda_k \mathbf{c}^T\mathbf{x}^k = \sum_{k=1}^{p} \lambda_k z_0 = z_0.$$

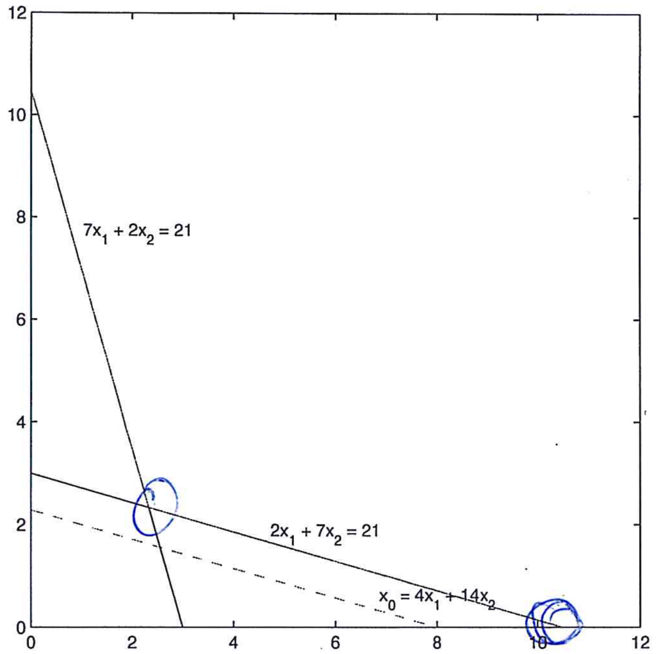


Figure 4.4. *Infinitely many Optimal Solutions*

Any $(x_1, x_2) = (1, k)$ for k being any positive number is a feasible solution.

Example 4.4. Consider

$$\begin{array}{ll} \max & x_0 = 4x_1 + 14x_2 \\ \text{subject to} & \begin{cases} 2x_1 + 7x_2 \le 21 \\ 7x_1 + 2x_2 \le 21 \\ x_1, x_2 \ge 0 \end{cases} \end{array}$$

	x_1	x_2	x_3	x_4	b
x_3	2	7*	1	0	21
x_4	7	2	0	1	21
x_0	−4	−14	0	0	0

↓

	x_1	x_2	x_3	x_4	b
x_2	2/7	1	1/7	0	3
x_4	45/7*	0	−2/7	1	15
x_0	$\boxed{0}$	0	2	0	42

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

↓↑

	x_1	x_2	x_3	x_4	b
x_2	0	1	7/45	−2/45	7/3
x_1	1	0	−2/45	7/45*	7/3
x_0	0	0	2	$\boxed{0}$	42

$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 7/3 \\ 0 \\ 0 \end{pmatrix}$

Thus all convex combinations of the points $[0, 3, 0, 15]$ and $[7/3, 7/3, 0, 0]$ are optimal feasible solutions.

4.4 Degeneracy and Cycling

Degenerate basic solutions are basic solutions with one or more basic variables at zero level. Degeneracy occurs when one or more of the constraints are redundant.

Example 4.5. Consider the following LLP

$$\max \quad x_0 = 2x_1 + x_2$$
$$\text{subject to} \begin{cases} 4x_1 + 3x_2 \leq 12 \\ 4x_1 + \;\, x_2 \leq 8 \\ 4x_1 - \;\, x_2 \leq 8 \\ \quad x_1,\, x_2 \geq 0 \end{cases}$$

	x_1	x_2	x_3	x_4	x_5	b
x_3	4	3	1	0	0	12
x_4	4**	1	0	1	0	8
x_5	4*	−1	0	0	1	8
x_0	−2	−1	0	0	0	0

(∗) ↙ ↘ (∗∗)

	x_1	x_2	x_3	x_4	x_5	b
x_3	0	4	1	0	−1	4
x_4	0	2*	0	1	−1	0
x_1	1	−1/4	0	0	1/4	2
x_0	0	−3/2	0	0	1/2	4

Degenerate Vertex { $x_4 = 0$ and basic }

↓

	x_1	x_2	x_3	x_4	x_5	b
x_3	0	2**	1	−1	0	4
x_1	1	1/4	0	1/4	0	2
x_5	0	−2	0	−1	1	0
x_0	0	−1/2	0	1/2	0	4

Degenerate Vertex { $x_5 = 0$ and basic }

↓

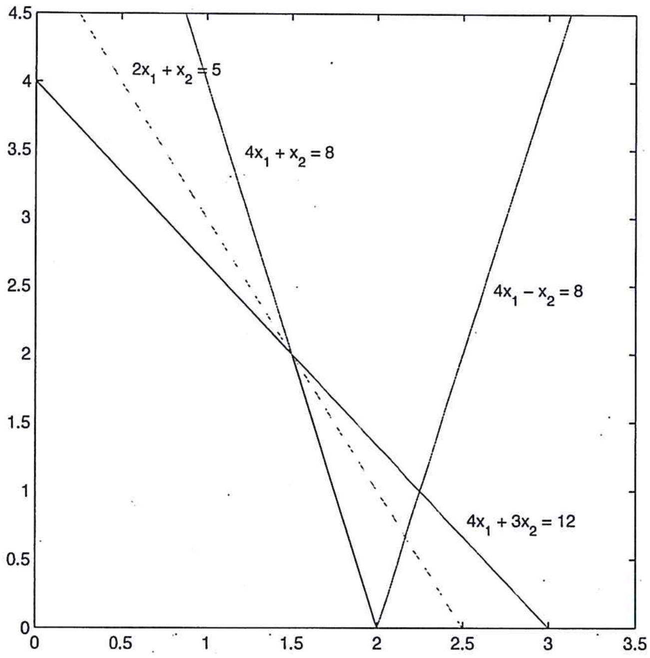


Figure 4.5. *Degeneracy and Cycling*

	x_1	x_2	x_3	x_4	x_5	b
x_3	0	0	1	-2	1^*	4
x_2	0	1	0	$1/2$	$-1/2$	0
x_1	1	0	0	$1/8$	$1/8$	2
x_0	0	0	0	$3/4$	$-1/4$	4

Degenerate Vertex { $x_2 = 0$ and basic }

↓

	x_1	x_2	x_3	x_4	x_5	b
x_5	0	0	1	-2	1	4
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_0	0	0	$1/4$	$1/4$	0	5

	x_1	x_2	x_3	x_4	x_5	b
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_5	0	0	1	-2	1	4
x_0	0	0	$1/4$	$1/4$	0	5

In figure 4.5, we see that the degenerate vertex V can be represented either by

$$\{x_2 = 0, x_4 = 0\}, \quad \{x_4 = 0, x_5 = 0\} \quad \text{or} \quad \{x_2 = 0, x_5 = 0\}.$$

We note that degeneracy guarantees the existence of more than one feasible pivot element, i.e. *tie-ratios* exist. For example, in the first tableau, the ratios for variables x_4 and x_5 are both equal to 2.

FCF $\rightarrow$ Starting BFS $\rightarrow$ Simplex Method $\rightarrow$ optimal solution

~~FCF~~

$$\max \ \vec{c}^T\vec{x} \qquad \text{s.t.} \begin{cases} A\vec{x} = \vec{b}\,(\geq \vec{0}) \\ \vec{x} \geq \vec{0} \end{cases}$$

$\xrightarrow{\text{artificial variables}}$

$$\max \ \vec{c}^T\vec{x} - \vec{M}^T\vec{x}_a \qquad \text{s.t.} \begin{cases} \left[A \mid I\right]\begin{bmatrix}\vec{x} \\ \vec{x}_a\end{bmatrix} = \vec{b} \\ \vec{x},\ \vec{x}_a \geq \vec{0} \end{cases}$$

$\downarrow$

starting BFS to augmented problem

$$\underbrace{\begin{pmatrix}\vec{x} \\ \vec{x}_a\end{pmatrix}}_{\text{basic variable}} = \begin{pmatrix}0 \\ \vec{b}\end{pmatrix} \geq \vec{0}$$

$\xleftarrow{\text{Simplex Method}}$ $(\vec{x}_a = \vec{0})$ $\leftarrow$ starting BFS to original problem

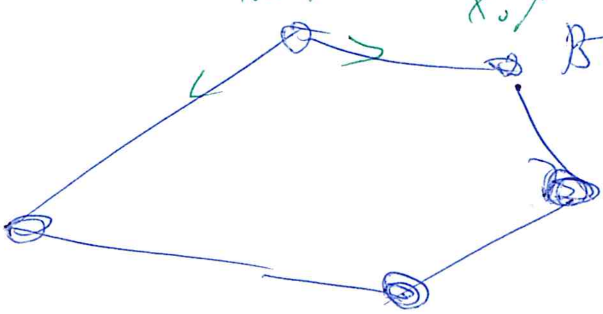
once x_a is outside basis (i.e. non basic)

$\Rightarrow x_a = 0$ for all later tableaux.

$x_a = 0$	$\longrightarrow$	$x_a > 0$
nonbasic		basic
Cost $-Mx_a = 0$		$-Mx_a\ (= -\infty)$

$x_0 \nearrow$

$x_0 \nearrow$



BFS (corner pt)

$x_0 \nearrow$

optimality condition

directional derivative

$x_0 \nearrow$

$$\text{FCF} \Rightarrow \text{FR} \neq \phi$$

$$\vec{0} \in \text{FR}$$

$\text{FR} = \phi$ $\vec{0} \notin \text{FR}$ $\cancel{\text{FCF}}$

$\Downarrow$

M-method add artificial variables

Simplex method

all $\vec{x}_a = \vec{0}$

starting BFS for original problem

$\Downarrow$

$\text{FR} \neq \phi$.

$\exists\, x_a \neq 0$

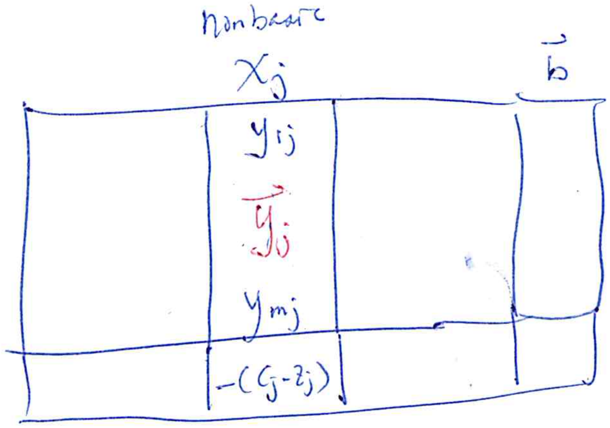
Simplex tableau reaches maximum

$\nexists$ BFS for original problem

$\Downarrow$

$\text{FR} = \phi$.

Tableau:



Y

Fact 1: $y_{1j}, y_{2j}, \dots, y_{mj} \le 0 \Rightarrow$ FR is unbdd in x_j-direction

Fact 2 In addition $-(c_j - z_j) < 0 \Rightarrow$ optimal x_0-solution is also unbdded

Fact 1

Pf: (i) Max $\vec{c}^T \vec{x}$

$$\begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \ge 0 \end{cases}$$

$$\boxed{A = BY}$$

$\vec{a}_j = B\vec{y}_j$ (i) $\quad \vec{y}_j \le \vec{0}$ (ii)

(ii) $A\vec{x} = \vec{b}$

$$[B|R]\begin{bmatrix} \vec{x}_B \\ \vec{0} \end{bmatrix} = \vec{b} \Rightarrow B\vec{x}_B = \vec{b} \quad \text{(iii)}$$

($\vec{x}_B \ge 0$)

$$\vec{b} = B\vec{x}_B = B\vec{x}_B - \theta\vec{a}_j + \theta\vec{a}_j \quad \forall \theta > 0$$

$$\overset{(i)}{=} B\vec{x}_B - \theta B\vec{y}_j + \theta\vec{a}_j \quad \forall \theta > 0$$

$$\vec{b} = B(\vec{x}_B - \theta\vec{y}_j) + \theta\vec{a}_j \quad \forall \theta > 0$$

$$\vec{b} = \underbrace{\begin{array}{c} \overbrace{\quad}^{m} \ \overbrace{\quad}^{n-m} \\ m\left\{\begin{bmatrix} B & R \end{bmatrix}\right. \end{array}}_{A} \begin{bmatrix} \vec{X}_B - \theta \vec{y}_j \\ 0 \\ \vdots \\ 0 \\ \theta \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \}\, m \\ \\ \\ \\ \leftarrow j^{th} \text{ entry} \\ \\ \\ \\ \end{array}$$

$$\| $$

$$m\{\underbrace{[\overbrace{B}^{m} | R]}_{A} \begin{bmatrix} \vec{X}_B \\ 0 \end{bmatrix} \begin{array}{l} \}\, m \\ \}\, n-m \end{array}$$

$$\vec{b} = A \underbrace{\begin{bmatrix} \overset{\geq 0}{\vec{X}_B - \theta y_j} \\ 0 \\ \vdots \\ 0 \\ \theta \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{x}_\theta \geq \vec{0}} \begin{array}{l} \}\, m \\ \\ \\ \\ \leftarrow j^{th} \text{ entry} \quad \theta > 0 \\ \\ \\ \\ \end{array}$$

$\vec{b} = A\vec{x}_\theta$, $\vec{x}_\theta \geq 0$

$\Rightarrow \quad \vec{x}_\theta \in FR, \quad \forall \theta > 0$

$\theta \to \infty \quad (\vec{x}_\theta)_j \to \infty$. FR is unbdd in x_j-direction

#

Fact 2: If $c_j - z_j > 0 \Rightarrow x_0 \uparrow \infty$

Pf: Current $x_0 = \vec{C}^T \vec{X}_{current}$ $\qquad \vec{X}_{current} = \begin{pmatrix} \vec{X}_B \\ 0 \end{pmatrix}$

$$= (\vec{C}_B^T, \vec{C}_R) \begin{pmatrix} \vec{X}_B \\ 0 \end{pmatrix} = \vec{C}_B^T \vec{X}_B$$

※

$$A = [\overbrace{B}^{m} \mid \overbrace{R}^{n-m}] \}m \qquad A = BY$$

$$\underset{1\times n}{\vec{C}^T} = [\underbrace{\vec{C}_B^T}_{m} \mid \underbrace{\vec{C}_R^T}_{n-m}] \qquad B_{m\times m} \quad Y_{m\times n}$$

$$\boxed{\underset{1\times n}{\vec{Z}^T} = \underset{1\times m}{\vec{C}_B^T} \, \underset{m\times n}{Y}}$$

$$\text{new } x_0 = \vec{C}^T \vec{X}_{new} = \vec{C}^T \underbrace{\begin{pmatrix} \vec{X}_B - \theta \vec{y}_j \\ \hline 0 \\ \vdots \\ 0 \\ \hline \theta \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}}_{X_\theta} \in FR$$

$$= (\vec{C}_B^T \mid \vec{C}_R^T)$$

$$= (\vec{C}_B^T \mid \cdots c_j \cdots)$$

$$= \vec{C}_B^T (\vec{X}_B - \theta \vec{y}_j) + c_j \theta$$

$$= \underbrace{\vec{C}_B^T \vec{X}_B} - \theta \, \underline{\vec{C}_B^T \vec{y}_j} + \theta c_j$$

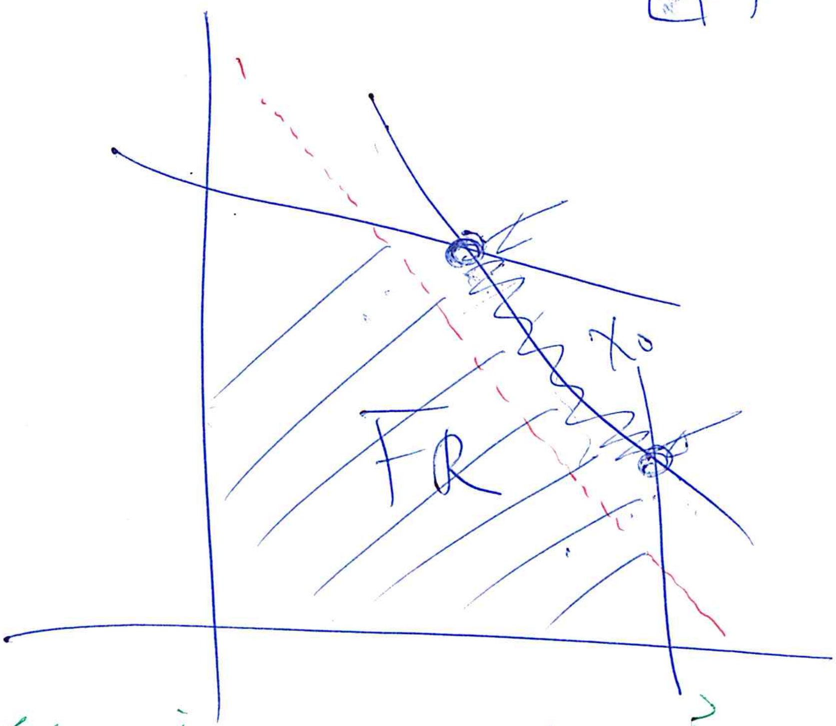
$$= \underset{\text{current}}{X_0} - \theta z_j + \theta c_j$$

$$= X_0 + \theta (c_j - z_j) \quad \forall \theta > 0.$$

$\theta \to \infty$ $\qquad \downarrow$ ∞ $\quad (\because c_j - z_j > 0)$

Ad

LPP



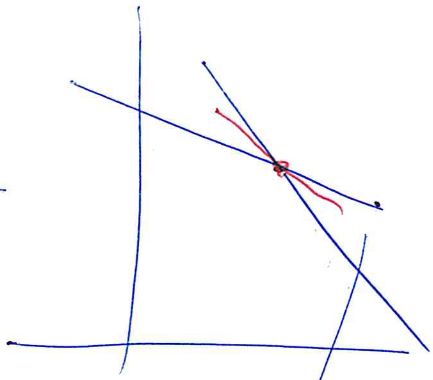
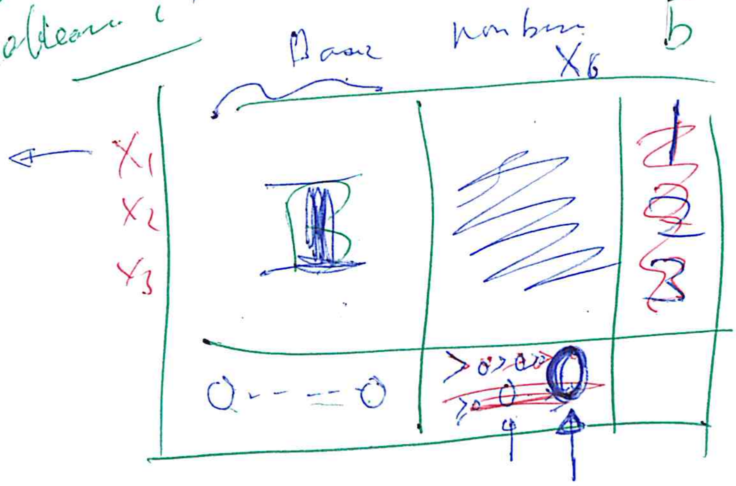


Tableau i



$$\begin{array}{l} \text{base} \\ \text{nonbase} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$$

$* \neq 0$ dependent

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in FR \longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_8 \end{pmatrix} = \begin{pmatrix} 0 \\ * \\ * \\ 0 \\ 0 \\ * \end{pmatrix} \in FR$$

$$X_0 \longrightarrow X_0$$