Assume $\vec{x}_0$ is optimal $\Rightarrow$ Construct $\vec{u}_0 \in FR_{Dual}$ s.t.

$$\underline{\vec{C}^T\vec{x}_0 = \vec{b}^T\vec{u}_0}$$

$\Downarrow$ W.D.T

$\underline{x_0, u_0 \text{ are optimal}}$

---

Assume $\vec{x}_0$ is $\underline{\text{optimal}}$

$$\Rightarrow \vec{Z} - \begin{pmatrix} \vec{c} \\ \vec{c}_s \end{pmatrix} \geq \vec{0}$$

$$\Rightarrow Y^T\vec{C}_B - \begin{pmatrix} \vec{c} \\ \vec{c}_s \end{pmatrix} \geq \vec{0}$$

$$\Rightarrow \begin{bmatrix} A^T \\ \hline I \end{bmatrix} (B^{-1})^T \vec{C}_B - \begin{pmatrix} \vec{c} \\ \vec{c}_s \end{pmatrix} \geq \vec{0} \qquad B^{-T} = (B^{-1})^T$$

$$\Rightarrow \underbrace{\begin{array}{l} \vec{Z}_x \{ \\ \vec{Z}_{x_s} \{ \end{array} \begin{bmatrix} A^T B^{-T} \vec{C}_B \\ \hline B^{-T} \cdot \vec{C}_B \end{bmatrix}}_{\vec{Z}} - \begin{pmatrix} \vec{c} \\ \vec{c}_s \end{pmatrix} \geq 0$$

$$\Rightarrow \begin{cases} A^T \underbrace{(B^{-T}\vec{c}_B)}_{\vec{u}_0} \geq \vec{c} & (1) \\ \underbrace{(B^{-T}\vec{C}_B)}_{\vec{u}_0} \geq \vec{c}_s = 0 \end{cases} \qquad \left( \vec{u}_0 \triangleq B^{-T}\vec{C}_B \right)$$

$$\Rightarrow \begin{cases} A^T\vec{u}_0 \geq \vec{c} \\ \vec{u}_0 \geq \vec{0} \end{cases} \Rightarrow \vec{u}_0 \in FR_{Dual}$$

$$\vec{b}^T \vec{\pi}_0 = \vec{b}^T (B^{-T}) \vec{c}_B$$

$$= \vec{c}_B^{\,T} B^{-1} \vec{b}$$

$$= \vec{c}_B^{\,T} \vec{x}_B = (\vec{c}_B, \vec{c}_R) \begin{pmatrix} \vec{x}_B \\ 0 \end{pmatrix}$$

$$= \underline{\vec{c}^{\,T} \cdot \vec{x}_0}$$

#

W.D.T $\Rightarrow$ $\vec{\pi}_0$ is also optimal #

P3



Since $x_{n+j}$ are slack variables, the corresponding columns in $A$ are just the $j$th unit vector $\mathbf{e}_j$ and the corresponding cost coefficients $c_{n+j} = 0$. Thus

$$\mathbf{e}_j^T (B^{-1})^T \mathbf{c}_B \geq c_{n+j} = 0, \quad j = 1, 2, \cdots, s.$$

Hence $\mathbf{u}_0 = (B^{-1})^T \mathbf{c}_B \geq \mathbf{0}$. Finally, since

$$\mathbf{b}^T \mathbf{u}_0 = \mathbf{b}^T (B^{-1})^T \mathbf{c}_B = \mathbf{c}_B^T B^{-1} \mathbf{b} = \mathbf{c}^T \mathbf{x}_0,$$

we see that $\mathbf{u}_0$ satisfies (5.2) and by Theorem 5.3, it is an optimal solution to the dual problem. □

This theorem also give us an explicit form of the optimal solution to the dual problem

**Theorem 5.5.** *If $B$ is the basis matrix for the primal corresponding to an optimal solution and $\mathbf{c}_B$ contains the prices of the variables in the basis, then an optimal solution to the dual is given by $(B^{-1})^T \mathbf{c}_B$, i.e., the entries in the $x_0$ row under the columns corresponding to the slack variables give the values of the dual structural variables. Moreover, the entries in the $x_0$ row under the columns for the structural variables will give the optimal values of the dual surplus variables.*

*Proof.* We only have to prove that $\mathbf{u}_B \equiv (B^{-1})^T \mathbf{c}_B$ is given by the entries in the $x_0$ row under the columns corresponding to the slack variables. In fact, for the slack variables, we have

$$z_{n+j} - c_{n+j} = z_{n+j} = \mathbf{e}_j^T (B^{-1})^T \mathbf{c}_B = \mathbf{u}_{B_j}.$$

Next we prove that the entries in the $x_0$ row under the columns for the structural variables give the optimal values of the dual surplus variables. Since

$$z_j = \mathbf{c}_B^T \mathbf{y}_j = \mathbf{c}_B^T (B^{-1} \mathbf{a}_j) = \mathbf{a}_j^T B^{-T} \mathbf{c}_B = \mathbf{a}_j^T \mathbf{u}_B, \quad j = 1, \cdots n,$$

and $A^T \mathbf{u}_B - \mathbf{u}_{B_s} = \mathbf{c}$ where $\mathbf{u}_{B_s}$ is the vector of dual surplus variables, we have

$$\mathbf{z} - \mathbf{c} = A^T \mathbf{u}_B - \mathbf{c} = \mathbf{u}_{B_s}.$$

□

*Example* 5.5. Let the primal problem be

$$\max \quad x_0 = 4x_1 + 3x_2 \qquad (4,3) = c$$

$$\text{subject to} \begin{cases} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 6 \\ 8 \\ 7 \\ 15 \\ 1 \end{bmatrix}}_{\vec{b}} \\ x_1, x_2 \geq 0. \end{cases}$$

$\vec{c}\ ,\ \vec{c}_s$

$(\vec{c}, \vec{c}_s) = (4,3,0,0,0,0,0)$

Standardizing the problem, we have

$$\begin{array}{ccccccc} \vec{a}_1 & \vec{a}_2 & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 & \vec{e}_5 \end{array}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{A \qquad\qquad I} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \\ 8 \\ 7 \\ 15 \\ 1 \end{bmatrix}}_{\vec{b}} \qquad \vec{X},\ \vec{X}_s$$

$$\begin{array}{ccccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \cdots & \vec{a}_7 \end{array}$$

$\vec{c} = (4\ 3\ 0\ 0\ 0\ 0\ 0)$

The optimal tableau is given by

P4

$B = (\vec{a}_3 \mid \vec{a}_2 \mid \vec{a}_4 \mid \vec{a}_1 \mid \vec{a}_7)$ $\quad B \cdot Y = [A \mid I]$

$\vec{c}_B = (c_3, c_2, c_4, c_1, c_7)$



$(4\,3\,0\,0\,0\,0\,0)$

$X_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_7 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 5 \\ 0 \\ 0 \\ 4 \end{pmatrix}$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | **b** |
|---|---|---|---|---|---|---|---|---|
| $x_3$ | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 2 |
| $x_2$ | 0 | 1 | 0 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ | 0 | 3 |
| $x_4$ | 0 | 0 | 0 | 1 | $\frac{3}{2}$ | $\frac{1}{2}$ | 0 | 5 |
| $x_1$ | 1 | 0 | 0 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 4 |
| $x_7$ | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 4 |
| $x_0$ | 0 | 0 | 0 | 0 | $\frac{5}{2}$ | $\frac{1}{2}$ | 0 | 25 |

$X_B$

$B = (\vec{e}_1, \vec{a}_2, \vec{e}_2, \vec{a}_1, \vec{e}_5)$

$\vec{c}_B = (0, 3, 0, 4, 0)$

$z^T - (\vec{c}^T, \vec{c}_s^T)$

Thus the optimal solution is $[x_1, x_2] = [4, 3]$ with $[x_3, x_4, x_5, x_6, x_7] = [2, 5, 0, 0, 4]$. From the $x_0$ row, we see that the optimal solution to the dual is given by

$$[u_1, u_2, u_3, u_4, u_5, u_6, u_7] = \left[0, 0, \frac{5}{2}, \frac{1}{2}, 0, 0, 0\right].$$

Let us verify this by considering the dual. The dual of the primal is given by

$$\min \quad u_0 = 6u_1 + 8u_2 + 7u_3 + 15u_4 + u_5$$

$$\text{subject to} \begin{cases} \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ u_i \geq 0, i = 1, 2, 3, 4, 5 \end{cases}$$

5 structural variable $u_1, \dots, u_5$

2 surplus variable $u_6, u_7$

Changing the minimization problem to a maximization problem and using simplex method (or the dual simplex method to be introduced in §4), we obtain the optimal tableau for the dual:

Structural Slup

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | **c** |
|---|---|---|---|---|---|---|---|---|
| $u_4$ | $\frac{1}{6}$ | $-\frac{1}{2}$ | 0 | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $u_3$ | $-\frac{1}{2}$ | $\frac{3}{2}$ | 1 | 0 | $-\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{3}{2}$ | $\frac{5}{2}$ |
| $u_0$ | 2 | 5 | 0 | 0 | 4 | 4 | 3 | $-25$ |

$U_0 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Thus the optimal solution for the dual is $[u_1, u_2, u_3, u_4, u_5] = \left[0, 0, \frac{5}{2}, \frac{1}{2}, 0\right]$ with optimal surplus variables $[u_6, u_7] = [0, 0]$. Notice that the optimal solution to the primal is given by the reduced cost coefficients for $u_4$ and $u_5$, i.e. $[x_1, x_2] = [4, 3]$ and the optimal values of the primal slack variables are given by $[x_3, x_4, x_5, x_6, x_7] = [2, 5, 0, 0, 4]$.

## 5.3 The Existence Theorem and The Complementary Slackness

**Theorem 5.6 (Existence Theorem).** *(i) An LPP has a finite optimal solution if and only if both it and its dual have feasible solutions.*