Solution to HW5

December 17, 2014

1. (i) The basic idea is to use $y_i = -x_i$ to replace x_i, which results in the familiar situation in which the dual simplex method can be applied. For simplicity, we still use x_i as variables, but the reader should keep in mind that the change of variable $y_i = -x_i$ has been done implicitly. With this in mind, the initial table is given by

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_3	-2	-2	-1	0	-8
x_4	-5	-3	0	-1	-15
x_0	120	100	0	0	0

Notice that the columns corresponding to basic variables have non-trivial entries being -1 rather than 1, this is because we keep the change of variables $y_i = -x_i$ on an implicit level rather than really spelling them out. It's quite clear from the table that x_4 is the departing variable. By computing ratios it's easy to see that the corresponding entering variavle is x_1. Basic column operations leads to the following table:

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_3	0	$-\frac{4}{5}$	-1	$\frac{2}{5}$	-2
x_1	-1	$-\frac{3}{5}$	0	$-\frac{1}{5}$	-3
x_0	0	28	0	-24	-360

Repeating the process described above we get the following final table:

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_2	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{2}$
x_0	0	0	35	10	430

Notice that we have scaled the constant -1 back to obtain the correct optimal value of z in the last entry of the above table. The reader should be able to figure out why the above table is also the final table by applying the simplex method, rather than dual simplex method.
(ii) The range for a change of b_1 must satisfy

$$\mathbf{x}_B + \Delta b_1 B^{-1} e_1 \geq 0.$$

Since we have calculated $\mathbf{x}_B = \left(\frac{5}{2}, \frac{3}{2}\right)$ and $B^{-1} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} \end{pmatrix}$, we get

$$\begin{cases} \frac{5}{2} + \Delta b_1 \left(\frac{5}{4}\right) & \geq 0, \\ \frac{3}{2} + \Delta b_1 \left(-\frac{3}{4}\right) & \geq 0, \end{cases}$$

which shows that $-2 \leq \Delta b_1 \leq 2$.
Similarly, $-3 \leq \Delta b_2 \leq 5$.
We have $\widehat{\mathbf{b}} = \left(\frac{7}{2}, \frac{1}{2}\right)^T$, then the new basic solution is given by

$$\begin{aligned} \widehat{\mathbf{x}}_B &= B^{-1}\widehat{\mathbf{b}} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 14 \end{pmatrix} \\ &= \begin{pmatrix} \frac{17}{4} \\ \frac{1}{4} \end{pmatrix}. \end{aligned}$$

Thus it is no longer feasible. The new objective value is

$$\mathbf{c}_B^T \widehat{\mathbf{x}}_B = (100, 120) \begin{pmatrix} \frac{17}{4} \\ \frac{1}{4} \end{pmatrix} = 455.$$

From this we can form the new starting stable:

	x_1	x_2	x_3	x_4	
x_2	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	$\frac{17}{4}$
x_1	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
	0	0	35	10	455

which is also the final table.

2. (i) Because the way to apply dual simplex method has been illustated in detail in Q1, I will not describe it here again. The final table is given by

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	0	0	1	$\frac{1}{2}$	0	0	3
x_6	0	0	0	$\frac{1}{2}$	1	1	9
x_1	1	2	0	0	1	0	6
	0	4	0	$\frac{1}{2}$	4	0	15

(ii) After the change of $\Delta\mathbf{c}$, the new $\mathbf{x}_0$ row is given by

$$\begin{aligned} \widehat{\mathbf{z}}^T - \widehat{\mathbf{c}}^T &= \widehat{\mathbf{c}}_B^T B^{-1} A - \widehat{\mathbf{c}}^T = \begin{bmatrix} -3 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & -3 & -\frac{3}{2} & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 2 & 4 & 0 \end{bmatrix}. \end{aligned}$$

As all the entries are non-negative, we conclue that the optimal solution remains unchanged.

3. (i) $z = \mathbf{c}^T\mathbf{x}^k = 1.2$.
(ii) The rescaled problem is:
minimize $\check{z} = -0.8w_1 - 0.1w_2 + 0.2w_3 + 1.9w_4$ subject to

$$\begin{cases} 0.8w_1 + 0.2w_3 & = 1, \\ 0.1w_2 + 1.9w_4 & = 2, \\ w_1, w_2, w_3, w_4 & \geq 0. \end{cases}$$

(iii) $\mathbf{d}_k$ is defined to be

$$\mathbf{d}_k = \frac{P_{A_k}\mathbf{c}_k}{\|P_{A_k}\mathbf{c}_k\|},$$

where $A_k = \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \\ 0 & 0.1 & 0 & 1.9 \end{bmatrix}$ and $P_A = I - A^T\left(AA^T\right)^{-1}A$. From this we get

$$d_k = \begin{pmatrix} 0.2156 \\ 0.4570 \\ -0.8626 \\ -0.0241 \end{pmatrix}.$$

(iv) The feasibility condition is given by $\mathbf{e} + \alpha\mathbf{d}_k \geq 0$. By definition of $\mathbf{e}$ and the value of $\mathbf{d}_k$ computed above, we can solve inequalities to get

$$\alpha \leq 1.1593.$$

Take $\alpha = 0.9\alpha_{max} = 1.0434$, subject it into z_{new}, we get

$$z_{new} = \begin{pmatrix} 1.225 \\ 1.4768 \\ 0.1 \\ 0.9749 \end{pmatrix}.$$

Back to the original LPP, we have $\mathbf{x}_{k+1} = \mathbf{x}z_{new}$, from which we get

$$\mathbf{x}_{k+1} = \begin{pmatrix} 0.98 \\ 0.1477 \\ 0.02 \\ 1.8523 \end{pmatrix}.$$

4. (i) The standard form is given by
maximize $z = 2x_1 - x_2$ subject to

$$\begin{cases} x_1 - x_2 + x_3 & = 15, \\ x_2 + x_4 & = 15, \\ x_1, x_2, x_3, x_4 & \geq 0. \end{cases}$$

(ii) Take $\mathbf{x}_0 = (16.9, 2, 0.1, 13)^T$, then $\mathbf{c} = (-2, 1, 0, 0)^T$ and $A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$, then

$$P_A\mathbf{c} = (-1, 0, 1, 0)^T.$$

From which we get

$$d = -\frac{P_A\mathbf{c}}{\|P_A\mathbf{c}\|} = \begin{pmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{pmatrix}.$$

The choice of α should satisfy the following feasibility condition:

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha\mathbf{d} \geq 0,$$

which forces α to satisfy $\alpha < \frac{0.1}{0.7071}$.
Take $\alpha = 0.9\alpha_{max} = \frac{0.09}{0.7071}$, we get

$$x_1 = x_0 + \alpha\mathbf{d} = \begin{pmatrix} 16.99 \\ 2 \\ 0.01 \\ 13 \end{pmatrix}$$

and $z_{max} = -31.98$.
Note The above is the new solution obtained without rescaling, which is the solution to an old version of this question. If one also includes the scaling of $\mathbf{x}_0$ to $\mathbf{e}$, then the correct new solution $\mathbf{x}_1$ to the LPP is given by

$$\mathbf{x}_1 = \begin{pmatrix} 15.0984 \\ 0.2 \\ 0.1016 \\ 14.8 \end{pmatrix}.$$

5.(i) $3x_1 + 4x_2 + 2x_3 - y^+ + y^- = 60$.
(ii) Let c^+ be the coefficient of y^+ and c^- the one for y^-, so $c^+ = 2c^-$.

6. (i) Minimize account under market share for Product 1 + account under market share for Product 2 subject to

$$\begin{cases} x_1 + x_2 + x_3 & \leq 55, \\ x_3 & \geq 10, \\ x_1, x_2 & \geq 0. \end{cases}$$

(ii) Let

$$y_1 = 0.5x_1 + 0.2x_3 - 15; y_1 = y_1^+ - y_1^-$$

$$y_2 = 0.3x_2 + 0.2x_3 - 10; y_2 = y_2^+ - y_2^-$$

The problem is
minimize $y_1^- + y_2^-$ subject to

$$\begin{cases} 0.5x_1 + 0.2x_3 - y_1^+ + y_1^- & = 15, \\ 0.3x_2 + 0.2x_3 - y_2^+ + y_2^- & = 10, \\ x_1 + x_2 + x_3 & \leq 55, \\ x_3 & \geq 10, \\ x_1, x_2, y_1^+, y_1^-, y_2^+, y_2^- & \geq 0. \end{cases}$$

(iii) Campaign 1=13.333, Campaign 2=0, and Campaign 3=41.667. In this case, the weighted sum of derivations is 1.667.