Solution to HW4

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1. Introduce artificial variables y_1 and y_2 we get the following auxiliary LPP:
maximize $z = x_1 - 2x_2 - 3x_3 - x_4 - x_5 + 2x_6 - My_1 - My_2$ subject to

$$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 + x_5 + y_1 & = 12, \\ x_1 + 2x_2 + x_3 + x_4 + 2x_5 + x_6 & = 18, \\ 3x_1 + 6x_2 + 2x_3 + x_4 + 3x_5 + y_2 & = 24, \\ x_1, \cdots, x_6, y_1, y_2 & \geq 0. \end{cases}$$

Using the above constraints to eliminate y_1 and y_2 in the expression of z we get

$$z+(1-4M)x_1+(6-8M)x_2+(5-4M)x_3+(3-2M)x_4+(5-4M)x_5 = 36-36M.$$

From this we can form the initial table:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
y_1	1	2	2	1	1	0	1	0	12
x_6	1	2	1	1	2	1	0	0	18
y_2	3	6	2	1	3	0	0	1	24
	$1-4M$	$6-8M$	$5-4M$	$3-2M$	$5-4M$	0	0	0	$-36M$

Using standard simplex method we finally get the following table:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
x_3	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	3
x_6	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	9
x_1	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	6
	0	4	0	$\frac{1}{2}$	4	0	$M-\frac{13}{4}$	$M+\frac{3}{4}$	15

2. Introduce the slack variables x_3 and x_4 to convert the LPP to its standard form:
maximize $z = 3x_1 + 2x_2$ subject to

$$\begin{cases} 2x_1 + x_2 + x_3 & = 2, \\ 3x_1 + 4x_2 - x_4 & = 12, \\ x_1, x_2, x_3, x_4 & \geq 0. \end{cases}$$

Then we introduce the artificial variable x_5 to get an auxiliary LPP:
maximize $\widehat{z} = 3x_1 + 2x_2 - Mx_5$ subject to

$$\begin{cases} 2x_1 + x_2 + x_3 & = 2, \\ 3x_1 + 4x_2 - x_4 + x_5 & = 12, \\ x_1, \cdots, x_5 & \geq 0. \end{cases}$$

Expressing $\widehat{z}$ using x_1, x_2, x_3, x_4 we get

$$\widehat{z} = (3M + 3)x_1 + (4M + 2)x_2 - Mx_4 - 12M.$$

From this we can form the initial table:

	x_1	x_2	x_3	x_4	x_5	
x_3	2	1	1	0	0	2
x_5	3	4	0	-1	1	12
	$-3M - 3$	$-4M - 2$	0	M	0	$-12M$

Applying standard simplex method, we get the following table:

	x_1	x_2	x_3	x_4	x_5	
x_2	2	1	1	0	0	2
x_5	-5	0	-4	-1	1	4
	$5M + 1$	0	$4M + 2$	M	0	$-4M + 4$

Since the artificial variable x_5 is still positive in the optimal solution, the original LPP does not admit any optimal solution.

3. Introduce slack variables x_1 and x_2 to convert the LPP to the following standard form:
maximize $z = 2x + 3y$ subject to

$$\begin{cases} x + 3y + x_1 & = 9, \\ 2x + 3y + x_2 & = 12, \\ x, y, x_1, y_1 & \geq 0. \end{cases}$$

Form the initial tables as follows:

	x	y	x_1	x_2	
x_1	1	3	1	0	9
x_2	2	3	0	1	12
	-2	-3	0	0	0

At this stage one should choose y as the entering variable, and the corresponding departing variable is x_2. By Gaussian elimination we get:

	x	y	x_1	x_2	
x_1	$\frac{1}{3}$	0	1	-$\frac{1}{3}$	5
y	$\frac{2}{3}$	1	0	$\frac{1}{3}$	4
	0	0	0	1	12

This table is the final one, so the solution is optimal. However, since $x_1 \neq 0$, we see that the LPP admits infinitely many optimal solutions.

4. Maximize $z = 3x_1 + 8x_2$ subject to

$$\begin{cases} x_1 + 2x_2 & \leq 6, \\ 2x_1 + 2x_2 & \leq 6, \\ x_1 + 4x_2 & \leq 8, \\ x_1 + 9x_2 & \geq 9, \\ x_1, x_2 & \geq 0. \end{cases}$$

5. Minimize $z' = \mathbf{b}^T\mathbf{w}$ subject to

$$\begin{cases} A^T\mathbf{w} & \geq \mathbf{c}, \\ B^T\mathbf{w} & = \mathbf{d}, \\ \mathbf{w} & \geq 0. \end{cases}$$

6. This is a simple application of the duality theorem, which says that if the dual LPP admits an optimal solution, then so is the primal LPP. Moreover, the optimal solution to the primal LPP is given in terms of that of dual LPP by $z_{max} = \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$. Then we simply do the direct calculation

$$\mathbf{b}^T\mathbf{w} = \begin{pmatrix} 12 & 21 & 8 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 & 5 & 0 & 3 \end{pmatrix}^T = 139$$

to see that the optimal solution to the primal LPP is 139.

7. We check that $\mathbf{y} = \left(\frac{2}{3}, 0, \frac{14}{3}\right)$ and $\mathbf{x} = \left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$ are feasible for their respective problems, and that they have the same value. Clearly, $\mathbf{y} \geq 0$ and $\mathbf{x} \geq 0$ Substituting $\mathbf{y}$ into the main constraints of the LPP,
we find $\frac{2}{3} + \frac{14}{3} \geq 2$, $-\frac{2}{3} + \frac{14}{3} \geq 4$, $\frac{4}{3} + \frac{14}{3} \geq 6$, and $\frac{2}{3} + \frac{14}{3} \geq 2$, so $\mathbf{y}$ is feasible. Similarly substituting x into the main constraints of the dual LPP, i.e. maximize $z = 2x_1 + 4x_2 + 6x_3 + 2x_4$ subject to

$$\begin{cases} x_1 - x_2 - 2x_3 + x_4 & \leq 1, \\ -2x_1 + x_2 + x_4 & \leq 2, \\ x_1 + x_2 + x_3 + x_4 & \leq 1, \\ x_1, x_2, x_3, x_4 & \geq 0, \end{cases}$$

we find $-\frac{1}{3}+\frac{4}{3}\leq 1$, $\frac{1}{3}\leq 2$, and $\frac{1}{3}+\frac{2}{3}\leq 2$, so $\mathbf{x}$ is feasible. The value of $\mathbf{y}$ is $\frac{16}{3}$, and the value of $\mathbf{x}$ is $\frac{16}{3}$. Since these are equal, both are optimal.