Solution to HW3

1. The extreme points are given by $(0,0)$, $(0,3)$, $(6,0)$, $(3,2)$. At the point $(0,0)$, $z = 0$; at the point $(0,3)$, $z = 9$; at the point $(6,0)$, $z = 12$; at the point $(3,2)$, $z = 12$. Therefore the optimal solutions are $(6,0)$ and $(3,2)$, $z_{max} = 12$.

2. (a) By introducing the slcak variables u and v it's easy to convert the LPP into the following standard form:

$$\begin{cases} 2x + 2y + u & = 8, \\ 5x + 3y + v & = 15, \\ x, y, u, v & \geq 0. \end{cases}$$

(b) In our case, $\mathbf{a}_1 = (2,5)^T$, $\mathbf{a}_2 = (2,3)^T$ and $\mathbf{a}_3 = (1,1)^T$. It then follows that by taking $\alpha_1 = 1, \alpha_2 = -3, \alpha_3 = 4$ we get

$$\sum_{i=1}^{3} \alpha_i \mathbf{a}_i = 0.$$

You may start with any feasible solution to move it to a basic solution by following the standard process. But for simplicity let me take the basic feasible solution $(\frac{3}{2}, \frac{5}{2}, 0, 0)^T$. In this case, the corresponding initial table is given by

	x	y	u	v	
u	2	2	1	0	8
v	5	3	0	1	15
	-120	-100	0	0	

,

and by Gauss elimination the final table is

	x	y	u	v	
x	1	0	$\star$	$\star$	$\frac{3}{2}$
y	0	1	$\star$	$\star$	$\frac{5}{2}$
	0	0	35	10	

,

this finishes checking the optimality condition.

3. In the original version of HW3, the feasible region of this LPP is unbounded, and the LPP does not admit an optimal solution. Because of this, any one who says that the LPP does not admit an optimal solution will also be treated as providing the correct solution to the problem. For the current version, introduce slack variables x_4, x_5 and x_6 to change the LPP to its standard form:
maximize $z = 2x_1 + 3x_2 - x_3$ subject to

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 & = 6, \\ x_1 - 3x_2 - 3x_3 + x_5 & = 10, \\ x_1 + x_3 + x_6 & = 5, \\ x_1, \cdots, x_6 & \geq 0. \end{cases}$$

Form the initial table:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	2	-1	1	0	0	6
x_5	1	-3	-3	0	1	0	10
x_6	1	0	1	0	0	1	5
	-2	-3	1	0	0	0	0

It's clear x_2 should be chosen as the entering variable. A θ-ratio computation shows that the departing variable is x_4. By Gauss elimination we get the following table:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	$\frac{1}{2}$	1	-$\frac{1}{2}$	$\frac{1}{2}$	0	0	3
x_5	$\frac{5}{2}$	0	-$\frac{9}{2}$	$\frac{5}{2}$	1	0	19
x_6	1	0	1	0	0	1	5
	-$\frac{1}{2}$	0	-$\frac{1}{2}$	$\frac{3}{2}$	0	0	9

Choose x_1 as the entering variable. By computing θ-ratio, one sees that the departing variable is x_6. This gives us the following table:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	0	1	-1	$\frac{1}{2}$	0	-$\frac{1}{2}$	$\frac{1}{2}$
x_5	0	0	-7	$\frac{5}{2}$	1	-$\frac{5}{2}$	$\frac{13}{2}$
x_1	1	0	1	0	0	1	5
	0	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{23}{2}$

It's easy to see this is the final table. Therefore we conclude that $z_{max} = \frac{23}{2}$ and the corresponding basic solution is $(5, \frac{1}{2}, 0)^T$.

4. First we need to introduce the slack variables x_5, x_6, x_7 to convert the problem to the following stanard LPP:
maximize $z = x_1 + 2x_2 + x_3 + x_4$ subject to

$$\begin{cases} 2x_1 + x_2 + 3x_3 + x_4 + x_5 & = 8, \\ 2x_1 + 3x_2 + 4x_4 + x_6 & = 12, \\ 3x_1 + x_2 + 2x_3 + x_7 & = 18, \\ x_1, \cdots, x_7 & \geq 0. \end{cases}$$

From these datum we can form the following table:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_5	2	1	3	1	1	0	0	0	8
x_6	2	3	0	4	0	1	0	0	12
x_7	3	1	2	0	0	0	1	0	18
	-1	-2	-1	-1	0	0	0	1	0

From the table one sees that one could choose x_2 as the entering variable. By computing the θ-ratios $\frac{b_1}{a_{12}} = 8$, $\frac{b_2}{a_{22}} = 4$, $\frac{b_3}{a_{32}} = 18$, we see that the corresponding departing variable is x_6. In this new setting,the basic solution to the equation $B\mathbf{x} = b$, where $B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix}$ is $(x_2, x_5, x_7)^T = (4, 4, 14)^T$. By Gauss elimination we get the following table:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_5	$\frac{4}{3}$	0	3	-$\frac{1}{3}$	1	-$\frac{1}{3}$	0	0	4
x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	0	4
x_7	$\frac{7}{3}$	0	2	-$\frac{4}{3}$	0	-$\frac{1}{3}$	1	0	14
	$\frac{1}{3}$	0	-1	$\frac{5}{3}$	0	$\frac{2}{3}$	0	1	8

Now we choose x_3 as the entering variable and computing the θ-ratios $\frac{b_1}{a_{13}} = \frac{4}{3}$, $\frac{b_3}{a_{33}} = 7$. Thus the corresponding departing variable is x_5. Similarly we get the following table from Gauss elimination:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_3	$\frac{4}{9}$	0	1	-$\frac{1}{9}$	$\frac{1}{3}$	-$\frac{1}{9}$	0	0	$\frac{4}{3}$
x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	0	4
x_7	$\frac{13}{9}$	0	0	-$\frac{10}{9}$	-$\frac{2}{3}$	-$\frac{1}{9}$	1	0	$\frac{34}{3}$
	$\frac{7}{9}$	0	0	$\frac{14}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	0	1	$\frac{28}{3}$

Since now all the coefficients in the objective row are non-negative, the solution $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T = (0, 4, \frac{4}{3}, 0, 0, 0, \frac{34}{3})^T$ is optimal and $z_{max} = \frac{28}{3}$.

5. By introducing slack variables x_5, x_6 and x_7 we rewrite the LPP in its standard form:
maximize $z = 5x_1 + 2x_2 + x_3 + x_4$ subject to

$$\begin{cases} 2x_1 + x_2 + x_3 + 2x_4 + x_5 & = 6, \\ 3x_1 + x_3 + x_6 & = 15, \\ 5x_1 + 4x_2 + x_4 + x_7 & = 24, \\ x_1, \cdots, x_7 & \geq 0. \end{cases}$$

From this we form the following table

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_5	2	1	1	2	1	0	0	0	6
x_6	3	0	1	0	0	1	0	0	15
x_7	5	4	0	1	0	0	1	0	24
	-5	-2	-1	-1	0	0	0	1	0

First we choose x_1 as the entering variable and compute the θ-ratios $\frac{b_1}{a_{11}} = 3$, $\frac{b_2}{a_{21}} = 5$, $\frac{b_3}{a_{31}} = \frac{24}{5}$, so the corresponding departing variable is x_5. From this and Gauss elimination we get the following new table:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	3
x_6	0	-$\frac{3}{2}$	-$\frac{1}{2}$	-3	-$\frac{3}{2}$	1	0	0	6
x_7	0	$\frac{3}{2}$	-$\frac{5}{2}$	-4	-$\frac{5}{2}$	0	1	0	9
	0	$\frac{1}{2}$	$\frac{3}{2}$	4	$\frac{5}{2}$	0	0	1	15

Since the coefficients in the objective row are now non-negative, the solution $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T = (3, 0, 0, 0, 0, 6, 9)^T$ is optimal and $z_{max} = 15$.

6. First we introduce slack variables x_3 and x_4 to rewrite the LPP into the following standrad form:
maximize $z = 4x_1 + x_2$ subject to

$$\begin{cases} 3x_1 + x_2 & = 3, \\ 4x_1 + 3x_2 - x_3 & = 6, \\ x_1 + 2x_2 + x_4 & = 3, \\ x_1, x_2, x_3, x_4 & \geq 0. \end{cases}$$

Then we introduce an artificial variable x_5 to further convert the problem to the following LPP:
maximize $\widehat{z} = 4x_1 + 2x_2 - Mx_5$ subject to

$$\begin{cases} 3x_1 + x_2 & = 3, \\ 4x_1 + 3x_2 - x_3 + x_5 & = 6, \\ x_1 + 2x_2 + x_4 & = 3, \\ x_1, \cdots, x_5 & \geq 0. \end{cases}$$

In fact, we can use the first equation $x_1 = 1 - \frac{x_2}{3}$ to further simplify the problem to the following LPP:
maximize $\widehat{z} = 4 - \frac{1}{3}x_2 - Mx_5$ subject to

$$\begin{cases} \frac{5}{3}x_2 - x_3 + x_5 & = 2, \\ \frac{5}{3}x_2 + x_4 & = 2, \\ x_2, x_3, x_4, x_5 & \geq 0. \end{cases}$$

We can express $\widehat{z}$ by non-basic variables:

$$\widehat{z} = (\frac{5}{3}M - \frac{1}{3})x_2 - Mx_3 + 4 - 2M.$$

Now one can form the initial table for the above LPP.

	x_2	x_3	x_4	x_5	$\widehat{z}$	
x_5	$\frac{5}{3}$	-1	0	1	0	2
x_4	$\frac{5}{3}$	0	1	0	0	2
	$\frac{1}{3} - \frac{5}{3}M$	M	0	0	1	$4 - 2M$

Since $M \gg 0$, we should choose x_2 as the entering variable. Compute the θ-ratio: $\frac{b_1}{a_{11}} = \frac{6}{5} = \frac{b_2}{a_{21}}$, by Bland's rule we choose x_5 as the departing variable. By Gauss elimination we get the following table:

	x_2	x_3	x_4	x_5	$\widehat{z}$	
x_2	1	$-\frac{3}{5}$	0	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	1	1	-1	0	0
	0	$\frac{1}{5}$	0	$M - \frac{1}{5}$	1	$\frac{18}{5}$

Since $M \gg 0$, it follows that the basic solution $(x_2, x_3, x_4, x_5)^T = (\frac{6}{5}, 0, 0, 0)^T$ is optimal and $\widehat{z}_{max} = \frac{18}{5}$. Since $M = 0$ in this case, $\widehat{z}_{max}$ is actually independent of x_5 and we have $z_{max} = \frac{18}{5}$.