Solution to HW2

the solution is for reference only

1. Let $D \subset \mathbb{R}^2$ denote the interior of the first quadrant in $\mathbb{R}^2$. More precisely,

$$D = \left\{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\right\}.$$

Take $E = \mathbb{R}^2 \setminus D$, then E is easily seen to be star shaped by taking the distinguished point x_0 to be the origin. However, E is not convex, the reason is as follows. Consider the points $x_1 = (0, 1)$ and $x_2 = (1, 0)$, then we have $x_1, x_2 \in E$. Let ℓ denote the line segment joining x_1 and x_2, it's clear that $\ell \not\subseteq E$, which implies E is not convex.

2. (a) Suppose a bounded convex subset $S \subset \mathbb{R}^n$ has a direction $\mathbf{v}$, then by definition there exists some $x_0 \in S$ such that

$$R := \{x_0 + \lambda \mathbf{v} | \lambda \geq 0\} \subset S,$$

but notice that R is unbounded, which implies that S is unbounded, a contradiction.

(b) There are two directions of S, determined respectively by the vectors

$$\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = (-1, 0).$$

(c) Consider $S = D$ with D being the first quadrant. Then S has a unique extreme point, which is the origin. Also notice that every $\mathbf{v}_\alpha = (1, \alpha)$ with $\alpha \geq 0$ is a direction of S.

3. We first show that S is a convex set. Take two points $(x_1, y_1), (x_2, y_2) \in S$, by definition of S this is equivalent to $y_1 \geq x_1^2$ and $y_2 \geq x_2^2$. Consider the point $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)$ with $0 < \lambda < 1$, which is just the point $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in \mathbb{R}^2$. Since

$$\begin{aligned} \lambda y_1 + (1 - \lambda)y_2 &\geq \lambda x_1^2 + (1 - \lambda)x_2^2 \\ &\geq (\lambda x_1 + (1 - \lambda)x_2)^2 \end{aligned}$$

by our assumption and elementary inequality, we see that $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in S$ for every $0 < \lambda < 1$. For every point $p \in \partial S$, consider the tangent line ℓ of ∂S passing through p, then it's clear by definition that ℓ is the supporting hyperplane of S at p. Since $x \geq 0$ and $y \geq 0$ for every point $(x, y) \in S$, S is a set bounded from below. By Theorem 1.5 in the lecture notes, ℓ contains an extreme point of S. Because $\ell \cap S = \{p\}$, p must be an extreme point of S. Since that the above argument works for every point of ∂S, we conclude that every point of ∂S is an extreme point of S.

4. Let $y_2 = -x_2$ and introduce slack variables y_0 and y_1 we get the following LPP of standard form.
Maximize $z = 3x_1 - 2y_2 - x_3 + x_4$ subject to

$$\begin{cases} x_1 - 2y_2 + x_3 - x_4 + y_0 & = 5, \\ -2x_1 + 4y_2 + x_3 + x_4 + y_1 & = -1, \\ x_1, x_2, x_3, x_4 & \geq 0, \\ y_0, y_1 & \geq 0. \end{cases}$$

5. Let $y_3 = 3 - x_3$, then by the restriction that $x_3 \leq 3$ we have $y_3 \geq 0$. The canonical form of the LPP is given as follows.
Maximize $z = -3x_1 - 2y_3$ subject to

$$\begin{cases} -x_1 + 2x_2 + y_3 & \leq 2, \\ x_1 - 2x_2 - y_3 & \leq -2, \\ -x_1 - x_2 & \leq -4, \\ x_1, x_2, y_3 & \geq 0. \end{cases}$$

6. (a) Simply sketch the feasible region, we see that the feasible set is bounded by four boundary components.

(b) It's easy to see from the constraints that we can take x and y to be sufficiently large provided that $x \geq y$. This is because when x and y are suffiently large then the sencond condition $x + 2y \geq 2$ is automatically satisfied. Since $x \geq y$, the first condition is also satisfied because $-3x + 2y \leq 0$. This shows that (x, y) with $x \geq y$ and $x, y \gg 0$ is always feasible. But such a choice will make z to be sufficiently large as the coefficients before x and y in z are both positive. This shows that there is no feasible solution of z.

7. (a) The extreme points are given by $(0, 3), (0, 1), (2, 0) \in \mathbb{R}^2$.
(b) Let S be the feasible set of the given LPP, consider a closed subset E of S bounded by the x-axis, y-axis, $y = -\frac{1}{2}x + 1$ and $3x + 5y = 15$. Then it's quite clear from the pircture that all points in $S \setminus E$ gives larger values of z than the points in E. Because of this, we can replace the unbounded feasible set S by the bounded set E. By the extreme point theorem, we only need to compute the values of z at all the extreme points and then compare them. When $(x, y) = (0, 3)$, $z = 15$; when $(x, y) = (0, 1)$, $z = 5$; when $(x, y) = (2, 0)$, $z = 6$; when $(x, y) = (5, 0)$, $z = 15$. From this we conclude that $z_{min} = 5$.

8. (a) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ be the columns of A. Notice that

$$\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 - 2\mathbf{a}_5 = 0,$$

so $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -1, \alpha_4 = -1, \alpha_5 = -2$. We compute

$$\frac{x_1}{\alpha_1} = 2, \frac{x_2}{\alpha_2} = 3, \frac{x_3}{\alpha_3} = -2, \frac{x_4}{\alpha_4} = -3, \frac{x_5}{\alpha_5} = \frac{3}{2}.$$

From this we see that $r = 1$. By definition of the new solution $\mathbf{x}'$ we get

$$\mathbf{x}' = (0, 1, 4, 5, 7)^T.$$

Repeat the above process once again, we get a basic feasible solution

$$\mathbf{x}_1 = (0, 0, 4, 6, 8)^T.$$

(b) Use the method of (a), we can move $\mathbf{x}_1$ to the basic feasible solution

$$\mathbf{x}_2 = (0, 4, 6, 0, 4)^T,$$

which is adjacent to $\mathbf{x}_1$ in the sense that it differs from $\mathbf{x}_1$ by only one basic variable.