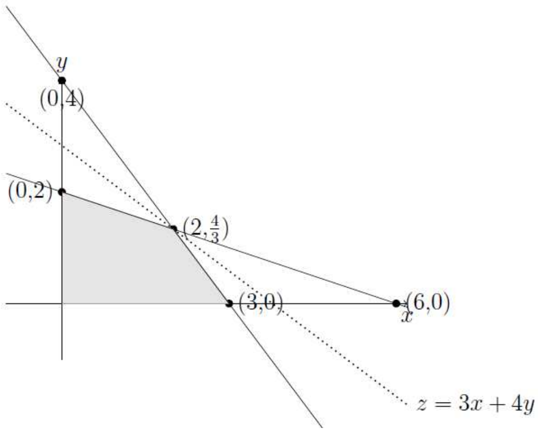
Solution to HW1

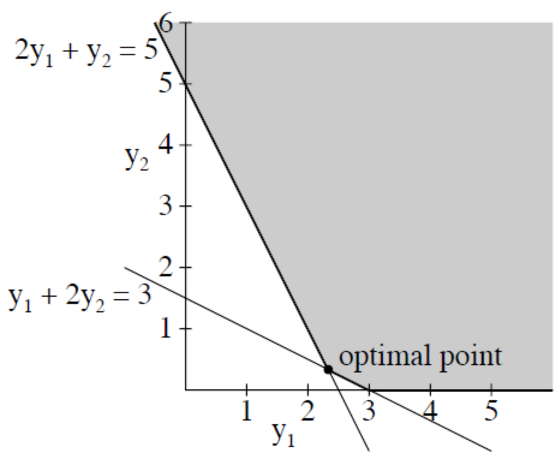
the solution is for reference only

1.



From the picture above, it's easy to see $z = 3x + 4y$ gets its maximal value at the point $(2, \frac{4}{3})$, by direct computation $z_{max} = \frac{34}{3}$.

2. The graph of the problem is:



The constraint set is shaded. The objective function, $y_1 + y_2$, has slope -1. As we move a line of slope -1 down, the last place it touches the constraint set is at the intersection of the two lines, $2y_1 + y_2 = 5$ and $y_1 + 2y_2 = 3$. The point of intersection, namely $(\frac{7}{3}, \frac{1}{3})$, is the optimal vector.

3. (a) By introducing the slack variables x_4, x_5 we can transform the problem to the following standard LPP:
maximize $z = 4x_1 + 2x_2 + 7x_3$ subject to

$$2x_1 - x_2 + 4x_3 + x_4 = 18$$

$$4x_1 + 2x_2 + 5x_3 + x_5 = 10$$

where $x_1, x_2, x_3, x_4, x_5 \geq 0$.

(b) As in the above problem, we want to find the basic solutions of $A\mathbf{x} = b$, where $A = \begin{bmatrix} 2 & -1 & 4 & 1 & 0 \\ 4 & 2 & 5 & 0 & 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$. The possible choices of B_i's are listed as follows:

$$B_1 = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}, B_3 = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix},$$
$$B_5 = \begin{bmatrix} -1 & 4 \\ 2 & 5 \end{bmatrix}, B_6 = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}, B_7 = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, B_8 = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix},$$
$$B_9 = \begin{bmatrix} 4 & 0 \\ 5 & 1 \end{bmatrix}, B_{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From these it's easy to compute the basic solutions with respect to these B_i's:

$$\mathbf{y}_1 = \begin{bmatrix} \frac{23}{4} \\ -\frac{13}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} -\frac{25}{3} \\ 0 \\ \frac{26}{3} \\ 0 \\ 0 \end{bmatrix}, \mathbf{y}_3 = \begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \\ 13 \\ 0 \end{bmatrix}, \mathbf{y}_4 = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \\ -26 \end{bmatrix}, \mathbf{y}_5 = \begin{bmatrix} 0 \\ -\frac{50}{13} \\ \frac{46}{13} \\ 0 \\ 0 \end{bmatrix},$$
$$\mathbf{y}_6 = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 23 \\ 0 \end{bmatrix}, \mathbf{y}_7 = \begin{bmatrix} 0 \\ -18 \\ 0 \\ 0 \\ 46 \end{bmatrix}, \mathbf{y}_8 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 9 \\ 0 \end{bmatrix}, \mathbf{y}_9 = \begin{bmatrix} 0 \\ 0 \\ \frac{9}{2} \\ 0 \\ -\frac{25}{2} \end{bmatrix}, \mathbf{y}_{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 18 \\ 10 \end{bmatrix}.$$

The basic variables are easily identified using the convention $\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$.

(c) By the restriction $x_1, x_2, x_3, x_4, x_5 \geq 0$, we only need to consider y_3, y_6, y_8, y_{10}. By direct computations we get (using the notations as in the above problem): $z_3 = 10, z_6 = 10, z_8 = 14, z_{10} = 0$. Since the set S of feasible solutions is nonempty and bounded, by the extreme point theorem we get $z_{\max} = z_4 = 14$.

4. Use $\mathbf{x}_1^*, \cdots, \mathbf{x}_n^*$ to denote the exteme points for which the optimal value of the objective function z is attained. Since z is a linear combination of the entries of $\mathbf{x} = (x_1, \cdots, x_k)^T$, we can write $z = \mathbf{c} \cdot \mathbf{x}, c \in \mathbb{R}^k$ and $z_{\max} = \mathbf{c} \cdot \mathbf{x}_i^*$ for all i. Let $\mathbf{y} = \sum_{i=1}^n d_i \mathbf{x}_i^*$ be a convex combination of $\mathbf{x}_i^*$'s, then it's necessary that $\sum_{i=1}^n d_i = 1$ and $d_i \geq 0$ for all i. We evaluate z at $\mathbf{y}$ to get:

$$z(\mathbf{y}) = \mathbf{c} \cdot \mathbf{y} = \mathbf{c} \sum_{i=1}^n d_i \mathbf{x}_i^* = \sum_{i=1}^n d_i \mathbf{c} \cdot \mathbf{x}_i^* = \mathbf{c} \cdot \mathbf{x}_i^* = z_{\max}.$$

This completes the proof.

5. *Proof.* Suppose that $\mathbf{x} = (x_1, \cdots, x_n) \in S$ is an extreme point of the canonical LPP. By adding slack variables $y_1, \cdots, y_m$ we can transform the LPP to its standard form, with the corresponding linear system given by $A\mathbf{x} = b$, where $\mathbf{x} \in \mathbb{R}^{m+n}$. Consider $(\mathbf{x}, y_1, \cdots, y_m) \in \mathbb{R}^{m+n}$, by definition, there exists $\mathbf{y} \in \mathbb{R}^m$ such that $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}$ satisfies the equation $A\mathbf{x} = b$, then it's clear that $(\mathbf{x}, \mathbf{y}) \in S'$. Then we only need to show that $(\mathbf{x}, \mathbf{y})$ is an extreme point of S'. In fact, suppose this is not the case, then there exists $\lambda \in (0, 1)$ such that $(\mathbf{x}, \mathbf{y}) = \lambda \mathbf{p}_1 + (1 - \lambda)\mathbf{p}_2$, where $\mathbf{p}_1, \mathbf{p}_2 \in S'$ are points different from $(\mathbf{x}, \mathbf{y})$. From this it follows that there exists $\mathbf{q}_1, \mathbf{q}_2 \in S$ distinct from $\mathbf{x}$ such that $\mathbf{x} = \lambda \mathbf{q}_1 + (1 - \lambda)\mathbf{q}_2$ with $\lambda \in (0, 1)$, which contradicts with the fact that $\mathbf{x}$ is an extreme point of the canonical LPP. This shows that every extreme point $\mathbf{x} \in S$ induces an extreme point $(\mathbf{x}, \mathbf{y}) \in S'$.
Conversely, suppose $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}$ is an extreme point of the standard LPP, in particular it satisfies the equation $A\mathbf{x} = b$. Since $\mathbf{y} \geq 0$ we deduce that $A\mathbf{x} \leq b$, so $\mathbf{x} \in S$. We only need to show $\mathbf{x}$ is an extreme point of the canonical LPP to complete the proof. Suppose on the contrary that $\mathbf{x}$ is not an extreme point so that there exists $\lambda \in (0, 1)$ such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$ with $\mathbf{x}_1, \mathbf{x}_2 \in S$ different from $\mathbf{x}$. This implies that $(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}_1, \mathbf{y}) + (1 - \lambda)(\mathbf{x}_2, \mathbf{y})$, which contradicts with the fact that $(\mathbf{x}, \mathbf{y}) \in S'$ is an extreme point. So for every extreme point $(\mathbf{x}, \mathbf{y}) \in S'$, its trucation $\mathbf{x} \in S$ is an extreme point.

6. *Proof.* We choose $x_1, x_2 \in f(S)$ and $\lambda \in [0, 1]$, then there exists $y_1, y_2 \in S$ such that $f(y_i) = x_i$ for $i = 1, 2$. Using the linearity of f we get $\lambda x_1 + (1 - \lambda)x_2 = \lambda f(y_1) + (1 - \lambda)f(y_2) = f\big(\lambda y_1 + (1 - \lambda)\big)$. By the convexity of S we have $\lambda y_1 + (1 - \lambda)y_2 \in S$, it follows that $\lambda x_1 + (1 - \lambda)x_2 \in f(S)$, which shows the convexity of $f(S)$.

7. (a) The first and third columns of A form the matrix $B_1 = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$, so the problem is to investigate the equation $B_1\mathbf{x} = \mathbf{b}$. Since $2\mathbf{b}$ gives the second column of B_1, it's easy to see the basic solution exists, which is $\mathbf{x}_{B_1} = (0, \frac{1}{2})^T$.

So the corresponding basic solutions of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_1 = (0, 0, \frac{1}{2}, 0, 0)^T$, which is degenerate by denition.

(b) Similarly, we have $B_2 = \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$. From the rst column of B_2 it's easy to see the basic solution of $B_2\mathbf{x} = \mathbf{b}$ exists, and is given by $\mathbf{x}_{B_2} = (\frac{2}{3}, 0)^T$. So the corresponding basic solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_2 = (0, \frac{2}{3}, 0, 0, 0)^T$, which is easily seen to be degenerate.

(c) $B_3 = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, again from the rst column of B_3 one sees that the basic solution of $B_3\mathbf{x} = \mathbf{b}$ exists, which is again $\mathbf{x}_{B_3} = \mathbf{x}_{B_2} = (\frac{2}{3}, 0)^T$. So the corresponding basic solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_3 = \mathbf{x}_2 = (0, \frac{2}{3}, 0, 0, 0)^T$, which is easily seen to be degenerate.