Solution to Midterm paper

1. *Proof.* It's a trivial fact that x is an extreme point of Q implies x is an extreme point of ∂Q.
Now assume x is an extreme point of ∂Q, we need to show that x cannot be written as $\lambda x_1 + (1-\lambda)x_2$ with $\lambda \in (0,1)$. One can argue by contradiction. Suppose $x_1, x_2 \in \partial Q$, then by assumption such a $\lambda \in (0,1)$ does not exist. Thus at least one x_i belongs to Q^{in}, where Q^{in} denotes the interior of Q. Without loss of generality, we may assume that $x_1 \in Q^{in}$. We claim that this implies $x \in Q^{in}$. In fact, if $x_2 \in Q^{in}$, then by the convexity of Q^{in}, $x \in Q^{in}$. So we may assume that $x_2 \in \partial Q$. At this stage we appeal to the expression $x = \lambda x_1 + (1-\lambda)x_2$, which implies that x_1 can be written as a linear combination of x and x_2. If $x \in \partial Q$, this implies that x_1 must lie on the line determined by x and x_2. Since Q is assumed to be a convex polytope, this is impossible. (Note that this is the only place where we use the assumption that Q is a polytope)
Now we have proved the existence of a $\lambda \in (0,1)$ such that $x = \lambda x_1 + (1-\lambda)x_2$ implies that $x \in Q^{in}$, but this contradicts with the assumption that x is an extreme point of ∂Q (so in particular $x \in \partial Q$). This completes the proof.

2. (a) To convert the LPP to its standard form, we introduce the slack variables x_4 and x_5, use $x_2 - 9$ to replace the original x_2, and use $-z$ to replace the original z. The answer is as follows.
Maximize $z = -3x_1 - 8x_2 - 4x_3$ subject to

$$\begin{cases} x_1 + x_2 - x_4 & = -1, \\ -2x_1 + 3x_2 - x_5 & = 27, \\ x_1, x_2, x_3, x_4, x_5 & \geq 0. \end{cases}$$

(b) We simply need to use $x_2 - 9$ to replace the original x_2.
Minimize $z = 3x_1 + 8x_2 + 4x_3$ subject to

$$\begin{cases} -x_1 - x_2 & \leq 1, \\ 2x_1 - 3x_2 & \leq -27, \\ x_1, x_2, x_3 & \geq 0. \end{cases}$$

3. We construct the LPP following the hint. The two rays ℓ_1, ℓ_2 are simply taken to be $y = 2x$ and $y = \frac{1}{2}x$ with $x \geq 0$. The feasible region F is bounded by ℓ_1 and ℓ_2 and taken to be

$$F = \left\{ (x,y) \in \mathbb{R}^2 | \frac{1}{2}x \leq y \leq 2x, x \geq 0 \right\}.$$

This determines the constraints of our LPP:

$$\begin{cases} 2x - y & \geq 0, \\ 2y - x & \geq 0, \\ x & \geq 0, \\ y & \geq 0. \end{cases}$$

It's easy to see the optimizer can be taken to be $z = x + y$, because the vector $\mathbf{v} = (1, 1)$ is a direction of the unbounded convex set F. So we end up with the following LPP, which does not admit an optimal solution.
Maximize $z = x + y$ subject to

$$\begin{cases} 2x - y & \geq 0, \\ 2y - x & \geq 0, \\ x & \geq 0, \\ y & \geq 0. \end{cases}$$

4. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ denote the colums of the matrix $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{bmatrix}$, then we see that

$$\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = 0,$$

i.e. $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = -1$. We compute

$$\frac{x_1}{\alpha_1} = 1, \frac{x_2}{\alpha_2} = \frac{1}{2},$$

from which we deduce $r = 2$. It then follows that

$$\widehat{x}_1 = x_1 - x_2 \frac{\alpha_1}{\alpha_2} = 1 - 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

$$\widehat{x}_2 = 0,$$

$$\widehat{x}_3 = x_3 - x_2 \frac{\alpha_3}{\alpha_2} = 2 + 1 \cdot \frac{1}{2} = \frac{5}{2}.$$

So $\widehat{\mathbf{x}} = (\frac{1}{2}, 0, \frac{5}{2})^T$.

5. First we convert it to a standard LPP by adding the slack variables x_4, x_5, x_6. The resulting LPP is:
maximize $z = 8x_1 + 9x_2 + 5x_3$ subject to

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 & = 2, \\ 2x_1 + 3x_2 + 4x_3 + x_5 & = 3, \\ 6x_1 + 6x_2 + 2x_3 + x_6 & = 8, \\ x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0. \end{cases}$$

The initial table is

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	2	1	0	0	2
x_5	2	3	4	0	1	0	3
x_6	6	6	2	0	0	1	8
	-8	-9	-5	0	0	0	0

It's clear from the table that the entering variable is x_2, and after computing the θ-ratios we see that the departing variable is x_5. Applying Gaussian elimination we get:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$\frac{1}{3}$	0	$\frac{2}{3}$	1	-$\frac{1}{3}$	0	1
x_2	$\frac{2}{3}$	1	$\frac{4}{3}$	0	$\frac{1}{3}$	0	1
x_6	2	0	-6	0	-2	1	2
	-2	0	7	0	3	0	9

At this stage we should choose x_1 as the entering variable. Again by computing θ-ratios we see that the corresponding departing variable is x_6. Using Gaussian elimination we get the following table.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	$\frac{5}{3}$	1	0	-$\frac{1}{6}$	$\frac{2}{3}$
x_2	0	1	$\frac{10}{3}$	0	1	-$\frac{1}{3}$	$\frac{1}{3}$
x_1	1	0	-3	0	-1	$\frac{1}{2}$	1
	0	0	1	0	1	1	11

It's clear that this is the final table, so $z_{max} = 11$ and the optimal solution is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (1, \frac{1}{3}, 0, \frac{2}{3}, 0, 0).$$

6. (i) We have

$$a_{i_j} = By_{i_j} = \sum_{k=1}^{n} y_{k,i_j} a_{i_k}.$$

Since $a_{i_1}, \cdots, a_{i_m}$ are linearly independent, we have

$$y_{j,i_j} = 1 \text{ and } y_{k,i_j} = 0 \text{ whenever } j \neq k,$$

which implies that $(y_{1,i_j}, \cdots, y_{m,i_j})^T$ are columns of I.
(ii) We argue by induction. Denote by B' and Y' the matrices corresponding to B and Y at the $i+1$-th step, then we have

$$y'_{rk} = \frac{y_{rk}}{y_{rj}}, \forall k = 1, \cdots, n.$$

And for each $i \neq r$,

$$y'_{ik} = y_{ik} - y_{ij} \cdot \frac{y_{rk}}{y_{rj}}, k = 1, \cdots, n.$$

Using the above one can compute

$$B'Y' = \mathbf{y}_k,$$

which is the k-th column of BY. Since the initial step of the induction is trivial, we are done.

(iii) This can again by argued by induction. With the same notation conventions as above, we have

$$x'_{ir} = \frac{x_{ir}}{y_{rj}},$$

and for any $k \neq r$,

$$x'_{ik} = x_{ik} - y_{kj} \cdot \frac{x_{ir}}{y_{rj}}.$$

Use this we have

$$\begin{aligned} A\mathbf{x}'_i &= \mathbf{x}'_{ir} \cdot \mathbf{a}_r + \sum_{k \neq r} \mathbf{x}'_{ik} \cdot \mathbf{a}_k \\ &= \mathbf{b}. \end{aligned}$$

Since for the initial table, we trivially have $A\mathbf{x}_i = \mathbf{b}$, the proof is complete.

(iv) Since

$$\begin{aligned} d_j &= c_j - z_j \\ &= c_j - \mathbf{c}_B^T \cdot \mathbf{y}_j, \end{aligned}$$

for $1 \leq j \leq m$ we have

$$\begin{aligned} d_{i_j} &= c_{j_j} - \mathbf{c}_B^T \cdot \mathbf{y}_{i_j} \\ &= c_{i_j} - c_{i_j} \\ &= 0. \end{aligned}$$