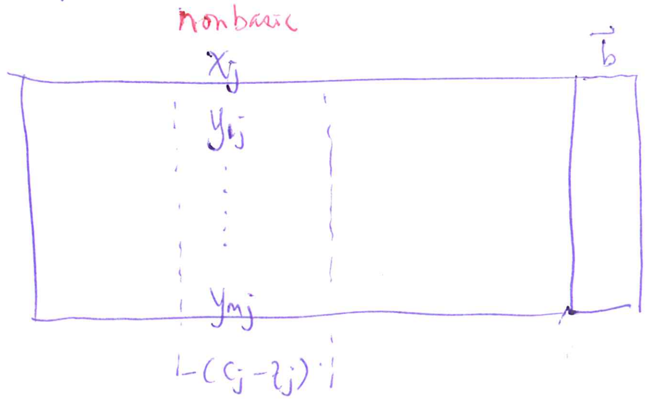
Consider a simplex tableau



<u>Fact 1</u>: If all $y_{ij} \le 0$ then FR is unbdded in x_j-direction.

Pf: Recall (i) $B\vec{x}_B = \vec{b}$ (ii) $A = BY$

$$B\vec{x}_B = \underbrace{[B \mid R]}_{A}\begin{bmatrix}\vec{x}_B \\ \vec{0}\end{bmatrix} \qquad\qquad A = BY \;\Downarrow\; \vec{a}_j = B\vec{y}_j \quad (\vec{y}_j \le \vec{0})$$

Thus:
$$\vec{b} = B\vec{x}_B = B\vec{x}_B - \theta\vec{a}_j + \theta\vec{a}_j \qquad \forall\,\theta > 0$$
$$= B\vec{x}_B - \theta B\vec{y}_j + \theta\vec{a}_j$$
$$= B(\vec{x}_B - \theta\vec{y}_j) + \theta\vec{a}_j$$

$$\Longleftrightarrow \quad \vec{b} = \underbrace{\left[\, B \;\middle|\; \underbrace{* * * \cdots * \;\vec{a}_j\; * \cdots *}_{R} \,\right]}_{A} \underbrace{\begin{array}{c} \begin{bmatrix} \overset{\ge 0}{\vec{x}_B - \theta\vec{y}_j} \\ \hline 0 \\ \vdots \\ 0 \\ \theta > 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \}\,m \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array}}_{\text{Feasible non-basic solution}} \qquad \theta > 0$$

FR contains solution $\begin{bmatrix} \vec{x}_B - \theta\vec{y}_j \\ 0 \\ \vdots \\ 0 \\ \theta \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ which is unbdded in x_j-direction.

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Fact 2 : If FURTHERMORE, $-(c_j - z_j) = z_j - c_j < 0$, then optimal solution is unbounded.

pf: (i) Current $x_0 = [\vec{c}_B^{\,T} \mid \vec{c}_R^{\,T}] \begin{bmatrix} \vec{x}_B \\ 0 \end{bmatrix} = \vec{c}_B^{\,T} \vec{x}_B$ (ii) $\vec{z}^{\,T} = \vec{c}_B Y \Rightarrow z_j = \vec{c}_B^{\,T} \vec{y}_j$

$$\text{new } x_0 = [\vec{c}_B^{\,T} \mid \underset{\substack{\uparrow \\ c_j}}{\vec{c}_R^{\,T}}] \underbrace{\begin{bmatrix} \vec{x}_B - \theta \vec{y}_j \\ \hline 0 \\ \vdots \\ 0 \\ \hline \theta \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}}_{\in FR \ \forall \theta > 0}$$

$$= \vec{c}_B^{\,T} \vec{x}_B - \theta \vec{c}_B^{\,T} \vec{y}_j + \theta c_j$$

$$\overset{(ii)}{=} \vec{c}_B^{\,T} \vec{x}_B - \theta z_j + \theta c_j$$

$$\overset{(i)}{=} x_0 + \theta (c_j - z_j)$$

letting $\theta \to \infty$ new $x_0 \to \infty$ #

Weak Duality Thm

Primal (LPP_p)

$$\max \vec{c}^T\vec{x}$$

$$FR_p \leftarrow \begin{cases} A\vec{x} \le \vec{b} \\ \vec{x} \ge 0 \end{cases}$$

Dual (LPP_d)

$$\min \vec{b}^T\vec{u}$$

$$FR_d \leftarrow \begin{cases} A^T\vec{u} \ge \vec{c} \\ \vec{u} \ge \vec{0} \end{cases}$$

(We don't assume $\vec{b} \ge \vec{0}$ or $\vec{c} \ge \vec{0}$)

Thm 5.2 $\quad \vec{x} \in FR_p,\ \vec{u} \in FR_d \Rightarrow \vec{c}^T\vec{x} \le \vec{b}^T\vec{u}$

Pf:
$$\begin{aligned} \vec{c}^T\vec{x} = \vec{x}^T\vec{c} &\le \vec{x}^T A^T\vec{u} \quad (\because A^T\vec{u} \ge \vec{c},\ \vec{x} \ge \vec{0}) \\ &= \vec{u}^T A\vec{x} \\ &\le \vec{u}^T\vec{b} \quad (\because A\vec{x} \le \vec{b},\ \vec{u} \ge \vec{0}) \quad \# \end{aligned}$$

Corollary (Thm 5.3) $\quad \vec{x}_0 \in FR_p,\ \vec{u}_0 \in FR_d$ s.t. $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$

$\Rightarrow$ $\vec{x}_0$ is optimal for LPP_p & $\vec{u}_0$ is optimal for LPP_d

Pf: $\quad \vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0 \underset{\text{Thm 5.2}}{\ge} \vec{c}^T\vec{x} \quad \forall \vec{x} \in FR_p \quad \#$

Strong Duality Thm (Converse of Thm 5.3)

Thm 5.4 $\quad \vec{x}_0 \in FR_p,\ \vec{u}_0 \in FR_d$ s.t. $\vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$

$\Longleftrightarrow$ $\vec{x}_0$ is optimal for LPP_p OR $\vec{u}_0$ is optimal for LPP_d.

First let's prove the "OR" of the theorem is true

Pf: $\vec{x}_0$ is optimal for $LPP_p \underset{\text{Thm 5.4}}{\Rightarrow} \vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0 \underset{\text{Thm 5.3}}{\Rightarrow} \vec{u}_0$ is optimal for LPP_d.

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Thus we only need to prove: $\vec{x}_0$ is optimal for $LPP_p \Rightarrow \vec{c}^T\vec{x}_0 = \vec{b}^T\vec{u}_0$

where $\vec{u}_0$ is optimal for LPP_d.

pf: primal

$$\max \vec{c}^T\vec{x} \quad \begin{cases} A\vec{x} \le \vec{b} \\ \vec{x} \ge \vec{0} \end{cases} \Rightarrow \max \vec{c}^T\vec{x} + \overset{\vec{0}}{\overset{\parallel}{\vec{c}_s}}{}^T\vec{x}_s \quad \begin{cases} A\vec{x} + I\vec{x}_s = \vec{b} \\ \vec{x}, \vec{x}_s \ge 0 \end{cases}$$

Since $\vec{x}_0$ is optimal for LPP_p $\Rightarrow$ $\vec{z} - \begin{pmatrix} \vec{c} \\ \underset{\overset{\parallel}{0}}{\vec{c}_s} \end{pmatrix} \ge \vec{0}$ (i)

Recall

(ii) $\vec{z}^T = \vec{c}_B^T Y$

(iii) $[A \mid I] = BY$

(iv) $\vec{b} = A\vec{x}_0 = [B|R]\begin{bmatrix} \vec{x}_B \\ 0 \end{bmatrix} = B\vec{x}_B \Rightarrow \vec{x}_B = B^{-1}\vec{b}$

(v) optimal value $= \vec{c}^T\vec{x}_0 = (\vec{c}_B, \vec{c}_R)\begin{pmatrix} \vec{x}_B \\ \vec{0} \end{pmatrix} = \vec{c}_B^T\vec{x}_B$

$\therefore$ $\vec{z} \overset{(ii)}{=} Y^T\vec{c}_B \overset{(iii)}{=} \begin{bmatrix} A^T \\ \hline I \end{bmatrix} B^{-T}\vec{c}_B$ $\quad (B^{-T} = (B^{-1})^T = (B^T)^{-1})$

$= \begin{bmatrix} A^T B^{-T}\vec{c}_B \\ \hline B^{-T}\vec{c}_B \end{bmatrix} \underset{(i)}{\ge} \begin{pmatrix} \vec{c} \\ \hline \vec{0} \end{pmatrix}$ (vi)

$\Rightarrow$ (a) $A^T(B^{-T}\vec{c}_B) \ge \vec{c}$ (b) $B^{-T}\vec{c}_B \ge \vec{0}$

Thus let $\vec{u}_0 = B^{-T}\vec{c}_B$ then $\begin{cases} A^T\vec{u}_0 \ge \vec{c} \\ \vec{u}_0 \ge 0 \end{cases} \Rightarrow \vec{u}_0 \in FR_d$

Moreover $\vec{b}^T\vec{u}_0 = \vec{b}^T B^{-T}\vec{c}_B = \vec{c}_B^T B^{-1}\vec{b} \overset{(iv)}{=} \vec{c}_B^T\vec{x}_B \overset{(v)}{=} \vec{c}^T\vec{x}_0$ ⋇

<u>Corollary</u>: The optimal solution of the dual problem, i.e. $\vec{u}_0$, is given as the reduced cost coefficient of the slack variables at the optimal tableau.

pf. From (vi), reduced cost coefficient of the slack variables

$= \vec{z}_s - \vec{c}_s^{\,=0}$

$= B^{-T}\vec{c}_B = \vec{u}_0$ ⋇