Analysis on block diagonal and triangular preconditioners for a PML system of an electromagnetic scattering problem

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ABSTRACT

We shall propose several block triangular preconditioners for a PML system of an electromagnetic wave scattering problem and analyze the spectral behavior of the preconditioned systems. When the PML system is discretized by edge element methods, it results in a discrete system with its stiffness matrix being complex, symmetric but indefinite, which can be formulated into a real symmetric but indefinite saddle-point system. In order to preserve the symmetry of the coefficient matrix, we present block triangular preconditioners with two-sided preconditioning for the discrete PML system. We will estimate the lower and upper bounds of positive and negative eigenvalues of the preconditioned matrices, respectively. On the other hand, one may also like to apply some iteration methods for nonsymmetric linear systems in applications although the discrete systems are symmetric. To this end, we propose a block triangular preconditioner to precondition the systems only from one side and analyze the spectrum of the preconditioned systems. In addition, we have also established a spectral estimate of the preconditioned system by an effective preconditioner that was recently developed in literature. Numerical experiments are presented to demonstrate the effectiveness and robustness of these new preconditioners and our theoretical predictions on the spectral bounds of the preconditioned systems.

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1. Introduction

Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain containing the origin with a boundary Γ_0, and Ω_0^c be the complement of its closure. In this work, we consider the following electromagnetic wave scattering problem by the impenetrable scatterer Ω_0 [1,2]:

$$\begin{cases} \boldsymbol{\nabla} \times (\mu^{-1} \nabla \times \boldsymbol{u}) - k^2 \varepsilon \boldsymbol{u} = \boldsymbol{0} & in \ \Omega_0^c, \\ \boldsymbol{u} \cdot \boldsymbol{\tau} = \boldsymbol{g} \cdot \boldsymbol{\tau} & on \ \Gamma_0, \end{cases} \quad (1.1)$$

where two operators $\boldsymbol{\nabla} \times$ and $\nabla \times$ are defined as

$$\boldsymbol{\nabla} \times w = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x} \right)^T, \quad \nabla \times \boldsymbol{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y},$$

with a scalar function w and a vector-valued function $\boldsymbol{v} = (v_1, v_2)^T$, respectively, $\boldsymbol{\tau}$ is unit tangential vector on Γ_0, μ and ε are the magnetic permeability and the electric permittivity, respectively, k is the wave number and $\boldsymbol{g}$ is the trace of a function

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$\tilde{\boldsymbol{g}} \in \boldsymbol{H}_{loc}(\mathbf{curl}; \overline{\Omega^c})$ on Γ_0, where $\boldsymbol{H}_{loc}(\mathbf{curl}; \overline{\Omega^c})$ is the set consisting of all functions on Ω^c whose restrictions to $\Omega^c \cap D$ belong to $H(\mathbf{curl}; \Omega^c \cap D)$ for any bounded domain D.

PML is a popular and effective technique developed first by Bérenger [3] to transform a PDE on an unbounded domain to another PDE on a bounded domain so that the approximate solution obtained on the bounded domain converges to the original solution exponentially in terms of some damping parameters. The PML has been extensively studied in the past two decades, see, e.g., [2,4–8]. The convergence of the PML scattering system for the time-dependent scattering problem (1.1) was established in [2], and the edge element discretized system of the PML system was studied in [1] and an effective preconditioner was proposed and analyzed there. Following the edge element discretization of the PML system of (1.1) in [1], we come to the following generalized saddle-point system:

$$\mathcal{M}z \equiv \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1.2}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ are symmetric but indefinite. However, little study can be found in literature on fast solvers for a generalized saddle-point system of form (1.2) when both matrices A and B are symmetric but indefinite. This is exactly our current case when the system (1.2) arises from the edge element discretization of a PML system of the electromagnetic wave scattering problem (1.1), and will be the major focus of this work. In most applications, the matrix A in a generalized saddle-point problem of form (1.2) corresponds to a second order differential operator and B to a first order differential operator, e.g., the problems from the discretization of Navier–Stokes and Maxwell equations [9–11]. But an important feature of the system (1.2) we shall investigate here is that the matrices A and B are obtained both from the discretization of second order differential operators respectively. Nearly no studies on preconditioning methods and their preconditioning effects for such generalized saddle-point systems are available. It was suggested in [12] to solve the finite-element PML system by the GMRES solver coupled with a strong approximate inverse preconditioner, and some optimal choices of the PML parameters were proposed and tested [13]. A moving PML sweeping preconditioner was introduced in [14] for solving the Helmholtz equation on a Cartesian finite difference grid, and it can dramatically reduce the number of GMRES iterations. Based on a crucial observation to its Schur complement, a symmetric positive definite block diagonal preconditioner $\mathcal{P}_s$ of the form

$$\mathcal{P}_s = \begin{pmatrix} \hat{A} & 0 \\ 0 & \hat{S} \end{pmatrix} \tag{1.3}$$

was proposed in [1] for the system (1.2), where $\hat{A}$ and $\hat{S}$ are two symmetric positive definite approximations of A and $S = A + BA^{-1}B$, respectively. This seems to be the only work in literature where preconditioning effects were analyzed mathematically for a PML edge finite element system. The preconditioning effects was analyzed in [1] under several general assumptions, and it is still rather difficult to verify all these abstract assumptions rigorously. In this work, we shall derive some explicit bounds for the positive and negative eigenvalues of the preconditioned system $\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}$, following a purely algebraic argument.

The symmetry of the coefficient matrix $\mathcal{M}$ in (1.2) motivates us to solve the system (1.2) by the MINRES. Nevertheless, the common bounds on the convergence rate of this method depend on the bounds on the eigenvalues of $\mathcal{M}$, including the lower bound of positive eigenvalues, and the upper bound of negative eigenvalues; e.g., see [15,16]. In order to preserve the symmetry of the coefficient matrix, we shall present some block triangular preconditioners with two-sided preconditioning for the system (1.2). And we estimate the lower and upper bound of positive and negative eigenvalues of the preconditioned matrices, respectively. On the other hand, one may also like to apply some iteration methods for nonsymmetric systems in applications, see, e.g., [17–23], although the discrete systems are symmetric. To this end, we propose another block triangular preconditioner to precondition the systems only from one side and analyze the spectrum of the preconditioned systems.

The organization of this paper is as follows. We give a brief description of how the generalized saddle-point system arises from the PML scattering problem in Section 2. In Section 3, we present block triangular preconditioners with symmetric preconditioning and estimate the lower and upper bounds of positive and negative eigenvalues of the preconditioned matrices, respectively. In Section 4, we establish another block triangular preconditioner with one-side preconditioning for the problem (1.2). In addition, we derive in Section 5 some explicit and sharp bounds for the positive and negative eigenvalues of the preconditioned system $\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}$. Numerical experiments are presented in Section 6 to show the effectiveness and robustness of these new preconditioners and our theoretical predictions on the spectral bounds of the preconditioned systems.

Throughout this paper, we use the following notation: for $H \in \mathbb{R}^{n \times n}$, we write $H > 0$ ($H \geq 0$) if H is symmetric positive definite (or semi-definite), and sp(H) for the spectrum of H. For $\lambda \in \mathbb{C}$, Re(λ) and Im(λ) denote the real and imaginary parts of λ, respectively. I is often used for a general identity matrix. As a matter of convenience, we introduce some basic spectral notations. For any symmetric indefinite matrix H, we assume sp(H) $\subset [-\lambda_H, -\gamma_H] \cup [\Gamma_H, \Lambda_H]$, with λ_H, γ_H, Γ_H and Λ_H being positive constants.

2. Generalized saddle-point system from the PML scattering problem

In [1,2], the authors have discussed the approximation of the electromagnetic wave scattering problem (1.1) on unbounded domain by PML technique, and the discretization of the PML variational system by the edge element method. In

this section, we will give a brief description of how the generalized saddle-point system (1.2) arises from the PML scattering problem.

The electromagnetic wave scattering problem (1.1) is defined on the unbounded domain Ω_0^c. The PML technique is an effective strategy to approximate Eq. (1.1) by a PML equation on a bounded domain. To this end, we define three domains with a sufficiently large positive constant L:

$$\Omega_1 = (-a,\ a)^2 \setminus \bar{\Omega}_0,\ \ \Omega_2 = (-b,\ b)^2 \setminus [-a,\ a]^2,\ \ \Omega_3 = (-L, L)^2 \setminus [-b,\ b]^2,$$

and set $\Omega_L = \Omega_1 \cup \Omega_2 \cup \Omega_3$ with Γ_L being its boundary. Then we introduce

$$H_{\boldsymbol{g}}(\mathbf{curl};\ \Omega_L) = \{\boldsymbol{u} = \boldsymbol{u}_r + \mathrm{i}\,\boldsymbol{u}_i\ :\ \boldsymbol{u}_r \in H_{\boldsymbol{g}_r}(\mathbf{curl};\ \Omega_L),\ \boldsymbol{u}_i \in H_{\boldsymbol{g}_i}(\mathbf{curl};\ \Omega_L)\},$$

where $\boldsymbol{g}_r$ and $\boldsymbol{g}_i$ are the real and imaginary parts of function $\boldsymbol{g}$, respectively, and (with $l = r$ or i)

$$H_{\boldsymbol{g}_l}(\mathbf{curl};\ \Omega_L) = \{\boldsymbol{v}\ :\ \boldsymbol{v} \in H(\mathbf{curl};\ \Omega_L),\ \boldsymbol{v} \cdot \boldsymbol{\tau}|_{\Gamma_L} = 0,\ \boldsymbol{v} \cdot \boldsymbol{\tau}|_{\Gamma_0} = \boldsymbol{g}_l \cdot \boldsymbol{\tau}|_{\Gamma_0}\}.$$

With the above preparations, the solution of the scattering problem (1.1) can be approximated by the PML solution $\boldsymbol{u}_L \in H_{\boldsymbol{g}}(\mathbf{curl};\ \Omega_L)$ of the following system [1,2]

$$\int_{\Omega_L} (\alpha + \mathrm{i}\beta)(\nabla \times \boldsymbol{u}_L)(\overline{\nabla \times \boldsymbol{\psi}})\mathrm{d}\boldsymbol{x} - \int_{\Omega_L} k^2((D + \mathrm{i}E)\boldsymbol{u}_L) \cdot \overline{\boldsymbol{\psi}}\mathrm{d}\boldsymbol{x} = 0\ \ \forall\,\boldsymbol{\psi} \in H_0(\mathbf{curl};\ \Omega_L)$$

where α and β are two given real constants, D and E are both two-by-two real diagonal matrices related to PML parameters [2]. By writing $\boldsymbol{u}_L = \boldsymbol{u}_r + \mathrm{i}\boldsymbol{u}_i$ in the above equation, we come to

$$a(\boldsymbol{u}_r, \boldsymbol{u}_i;\ \boldsymbol{\psi}_r, -\boldsymbol{\psi}_i) = \boldsymbol{0}\ \ \ \forall\,\boldsymbol{\psi}_r, \boldsymbol{\psi}_i \in H_0(\mathbf{curl};\ \Omega_L), \tag{2.1}$$

where the bilinear form $a(\boldsymbol{u}_r, \boldsymbol{u}_i;\ \boldsymbol{\psi}_r, -\boldsymbol{\psi}_i)$ is given by

$$\begin{aligned} a(\boldsymbol{u}_r, \boldsymbol{u}_i;\ \boldsymbol{\psi}_r, -\boldsymbol{\psi}_i) = &\int_{\Omega_L} \alpha[(\nabla \times \boldsymbol{u}_r)(\nabla \times \boldsymbol{\psi}_r) - (\nabla \times \boldsymbol{u}_i)(\nabla \times \boldsymbol{\psi}_i)]\mathrm{d}\boldsymbol{x} \\ &- \int_{\Omega_L} \beta[(\nabla \times \boldsymbol{u}_i)(\nabla \times \boldsymbol{\psi}_r) + (\nabla \times \boldsymbol{u}_r)(\nabla \times \boldsymbol{\psi}_i)]\mathrm{d}\boldsymbol{x} \\ &- \int_{\Omega_L} k^2(\boldsymbol{u}_r^T D\boldsymbol{\psi}_r - \boldsymbol{u}_i^T D\boldsymbol{\psi}_i)\mathrm{d}\boldsymbol{x} + \int_{\Omega_L} k^2(\boldsymbol{u}_i^T E\boldsymbol{\psi}_r + \boldsymbol{u}_r^T E\boldsymbol{\psi}_i)\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Next, we shall introduce the edge element discretization of the PML variational equation (2.1). Assume that Ω_L is covered by a quasi-uniform triangulation $\mathcal{T}_h$ of triangular elements, with h being the maximum diameter among all the triangles in $\mathcal{T}_h$. Let $\mathcal{E}_h$ be the set of all edges in the triangulation $\mathcal{T}_h$. Define

$$\begin{aligned} V_{h,\boldsymbol{g}_l}(\Omega_L) = \Big\{&\boldsymbol{v}\ :\ \boldsymbol{v} \in H(\mathbf{curl};\ \Omega_L),\ \boldsymbol{v}|_K \in \mathcal{R}_1,\ \forall K \in \mathcal{T}_h; \\ &\boldsymbol{v} \cdot \boldsymbol{\tau}|_e = 0,\ \forall e \in \mathcal{E}_h \cap \Gamma_L;\ \boldsymbol{v} \cdot \boldsymbol{\tau}|_e = \boldsymbol{g}_l \cdot \boldsymbol{\tau}|_e,\ \forall e \in \mathcal{E}_h \cap \Gamma_0\Big\}, \end{aligned}$$

with $l = r$ or i, and $\mathcal{R}_1$ being the space of linear polynomials as follows:

$$\mathcal{R}_1 = (\mathcal{P}_0)^2 \oplus \{\boldsymbol{p} \in (\tilde{\mathcal{P}}_1)^2\ :\ \boldsymbol{x} \cdot \boldsymbol{p} = 0\},$$

where $\mathcal{P}_0$ and $\tilde{\mathcal{P}}_1$ are the space of constants and the space of homogeneous linear polynomials, respectively. Similarly, we can define $V_{h,0}(\Omega_L)$. Then the edge element approximation of the variational equation (2.1) can be formulated as follows:

Find $\boldsymbol{u}_{r,h} \in V_{h,\boldsymbol{g}_r}(\Omega_L)$ and $\boldsymbol{u}_{i,h} \in V_{h,\boldsymbol{g}_i}(\Omega_L)$ such that

$$a(\boldsymbol{u}_{r,h}, \boldsymbol{u}_{i,h};\ \boldsymbol{\psi}_{r,h}, -\boldsymbol{\psi}_{i,h}) = \boldsymbol{0}\ \ \ \forall\,\boldsymbol{\psi}_{r,h}, \boldsymbol{\psi}_{i,h} \in V_{h,0}(\Omega_L). \tag{2.2}$$

The major task of this work is to propose some effective preconditioners for the use in an iterative method for solving the edge element system (2.2). For the purpose, we write the system in a matrix–vector form. Let m_h^0 be the number of all the edges in the triangulation $\mathcal{T}_h$ lying inside Ω_L and $(m_h - m_h^0)$ be the number of all the edges in $\mathcal{T}_h$ lying on $\partial\Omega_L$. Let $\{\boldsymbol{\phi}_h^j\}_{j=1}^{m_h}$ be the edge element basis functions of space $V_h(\Omega_L)$, then there exist $x = (x_1, x_2, \ldots, x_{m_h})^T$ and $y = (y_1, y_2, \ldots, y_{m_h})^T$ such that

$$\boldsymbol{u}_{r,h} = \sum_{j=1}^{m_h} x_j \boldsymbol{\phi}_h^j, \qquad \boldsymbol{u}_{i,h} = \sum_{j=1}^{m_h} y_j \boldsymbol{\phi}_h^j.$$

Now by substituting $\boldsymbol{u}_{r,h}$ and $\boldsymbol{u}_{i,h}$ above into Eq. (2.2), then keeping all the terms involving the first m_h^0 components of x and y on the left-hand side and moving the other terms to the right-hand side, we can rewrite equation (2.2) to the saddle-point system (1.2).

3. Block triangular preconditioners for symmetric preconditioning

As it is known, the convergence rate of the Krylov subspace methods, such as MINRES and GMRES, is closely related to the eigenvalues and the eigenvectors of the coefficient matrix in the concerned linear system [15–17,24]. But the eigenvalues of the matrices arising from many applications (like the matrix $\mathcal{M}$ in (1.2)) are not clustered. Therefore, many preconditioners have been developed, e.g., see [25–34], among which block triangular preconditioners are considered as one of the most popular ones. Block triangular preconditioners were applied for the solution of the Stokes problem, e.g., see [35–38], and their theoretical properties were studied in [39]. In this section, we propose and study the block triangular preconditioners of the following form for the more challenging system (1.2) arising from the PML system for electromagnetic scattering problems (see Section 2):

$$\mathcal{P} = \begin{pmatrix} \hat{L} & 0 \\ \eta\hat{L} & \hat{L} \end{pmatrix} \tag{3.1}$$

where $\hat{L}$ is a nonsingular matrix such that $\hat{A} =: \hat{L}\hat{L}^T$ is an approximation of A and η is a given constant. If we apply the Krylov subspace methods (like SYMMLQ or MINRES) to solve the system (1.2), we may not like to transform the original symmetric problem (1.2) into a nonsymmetric one. Therefore we shall consider the following preconditioned system:

$$\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}\hat{z} = \mathcal{P}^{-1}b, \quad \hat{z} = \mathcal{P}^T z.$$

As the preconditioner $\mathcal{P}$ is a block triangular matrix, the system $\hat{z} = \mathcal{P}^T z$ can be solved relatively easily. In addition, for judging the effectiveness of our new preconditioners $\mathcal{P}$, we shall derive the bounds for the eigenvalues of the preconditioned matrix $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$, including the lower bound of the positive eigenvalues and the upper bound of the negative eigenvalues; e.g., see [15,16]. For this purpose, we introduce three matrices

$$\tilde{A} = \hat{L}^{-1}A\hat{L}^{-T}, \quad \tilde{B} = \hat{L}^{-1}B\hat{L}^{-T}, \quad \tilde{S} = \hat{L}^{-1}S\hat{L}^{-T} = \tilde{A} + \tilde{B}\tilde{A}^{-1}\tilde{B}.$$

Although the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ is symmetric, the indefiniteness of the matrices A and B bring the great difficulty in the estimates of bounds on the spectra of $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$. The most existing studies for the generalized saddle-point system (1.2) were carried out only for the cases where matrix A or B is symmetric positive definite, and those analysis techniques do not apply to our current case. We shall introduce some new analysis techniques. To better follow and motivate our subsequent analysis, we first outline the basic steps, and introduce

$$\mathcal{A} =: \begin{pmatrix} \hat{L} & 0 \\ 0 & \hat{L} \end{pmatrix}. \tag{3.2}$$

Then we can write

$$\frac{z^T\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}z}{z^Tz} = \frac{z^T\mathcal{P}^{-1}\mathcal{A}(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})\mathcal{A}^T\mathcal{P}^{-T}z}{z^T\mathcal{P}^{-1}\mathcal{A}\mathcal{A}^T\mathcal{P}^{-T}z} \frac{z^T((\mathcal{A}^{-1}\mathcal{P})^T\mathcal{A}^{-1}\mathcal{P})^{-1}z}{z^Tz}, \tag{3.3}$$

from which we clearly see that some careful and sharp spectral estimates for the matrices $(\mathcal{A}^{-1}\mathcal{P})^T\mathcal{A}^{-1}\mathcal{P}$ and $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$ should be achieved in order to establish desired spectral estimates of the preconditioned matrix $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$. As it is seen, it is quite surprising that the eigenvalues of $(\mathcal{A}^{-1}\mathcal{P})^T\mathcal{A}^{-1}\mathcal{P}$ can be explicitly expressed, after some detailed analysis by taking full advantage of its special structure. But the spectral analysis on the matrix $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$ is much more complicated. Instead, for a general eigenvector (x, y) of $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$ in the same block form as in (1.2), we choose to consider directly the eigen-equations satisfied by x and y. We then write variable x in terms of variable y from the first equation and substitute x into the second one. Then after some technical and delicate derivations, we can establish a quadratic inequality for the corresponding eigenvalue, which enables us to get the desired estimate of the upper and lower bounds of the eigenvalues by solving the quadratic inequality.

It remains to establish the lower bound of the positive eigenvalues and the upper bound of the negative eigenvalue of the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$. For this aim, by using the relation (3.3) and the explicit bounds of eigenvalues of the system $(\mathcal{A}^{-1}\mathcal{P})^T\mathcal{A}^{-1}\mathcal{P}$, we need only to consider the extreme spectral bounds for the inverse of $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$. By Sherman–Morrison–Woodbury formula, we can find that $(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}$ has the same form as $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$. Thereby we can derive the lower and the upper bounds of the eigenvalues of $(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}$ in a similar argument as we did for $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$ earlier.

3.1. Spectral estimates of matrices $(\mathcal{A}^{-1}\mathcal{P})^T\mathcal{A}^{-1}\mathcal{P}$ and $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$

In this subsection, we first work out the explicit formula for the eigenvalues of $(\mathcal{A}^{-1}\mathcal{P})^T\mathcal{A}^{-1}\mathcal{P}$ and the spectral estimate of $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$ as it was motivated above. It follows from (3.2) that

$$(\mathcal{A}^{-1}\mathcal{P})^T\mathcal{A}^{-1}\mathcal{P} = \begin{pmatrix} (1+\eta^2)I & \eta I \\ \eta I & I \end{pmatrix},$$

which is similar to the block diagonal matrix

$$\text{diag}\left\{ \begin{pmatrix} 1+\eta^2\sigma_1^2 & \eta\sigma_1 \\ \eta\sigma_1 & 1 \end{pmatrix}, \begin{pmatrix} 1+\eta^2\sigma_2^2 & \eta\sigma_2 \\ \eta\sigma_2 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1+\eta^2\sigma_n^2 & \eta\sigma_n \\ \eta\sigma_n & 1 \end{pmatrix} \right\},$$

whose eigenvalues can be given explicitly by

$$\frac{2 + \eta^2 \pm \sqrt{4\eta^2 + \eta^4}}{2}.$$

Then we come to the following conclusion from the above analysis.

Lemma 3.1. *The matrix* $(\mathcal{A}^{-1}\mathcal{P})^T \mathcal{A}^{-1}\mathcal{P}$ *has only two distinct eigenvalues* $\frac{1}{2}(2 + \eta^2 + \sqrt{4\eta^2 + \eta^4})$ *and* $\frac{1}{2}(2 + \eta^2 - \sqrt{4\eta^2 + \eta^4})$.

In the following, we deduce the spectral estimate of the matrix $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$. To do so, we first derive the spectral bounds for a matrix of the following specific structure:

$$\mathcal{R} = \begin{pmatrix} W & T \\ T^T & -W \end{pmatrix}, \tag{3.4}$$

where $W \in \mathbb{R}^{n \times n}$ is a symmetric indefinite matrix, and $T \in \mathbb{R}^{n \times n}$.

Lemma 3.2. *The eigenvalues of the matrix* $\mathcal{R}$ *in* (3.4) *lie in the following range:*

$$\left[-\frac{\Lambda_W + \lambda_W + \sqrt{(\Lambda_W - \lambda_W)^2 + 4\|T\|_2^2}}{2}, \frac{\Lambda_W + \lambda_W + \sqrt{(\Lambda_W - \lambda_W)^2 + 4\|T\|_2^2}}{2} \right].$$

Proof. Let λ and $(x^T, y^T)^T$ be the eigenvalue and eigenvector of matrix $\mathcal{R}$, then we can write

$$Wx + Ty = \lambda x, \quad T^T x - Wy = \lambda y. \tag{3.5}$$

If $y = 0$, we see that $T^T x = 0$ and $Wx = \lambda x$, which imply $\lambda \in [-\lambda_W, \Lambda_W]$. This suggests us to consider only $\lambda \notin [-\lambda_W, \Lambda_W]$ for $y \neq 0$. Clearly $\lambda I - W$ is nonsingular, and it follows from the first equality of (3.5) that

$$x = (\lambda I - W)^{-1} Ty.$$

Substituting into the second equality in (3.5), we obtain

$$T^T (\lambda I - W)^{-1} Ty - Wy = \lambda y,$$

then it follows by multiplying the above equality from the left by $y^T / y^T y$ that

$$\lambda = \frac{y^T T^T (\lambda I - W)^{-1} Ty}{y^T y} - \frac{y^T Wy}{y^T y}. \tag{3.6}$$

Now we first consider the case that $\lambda > \Lambda_W$. Clearly $\lambda I - W \geq (\lambda - \Lambda_W) I > 0$, which implies that $0 < (\lambda I - W)^{-1} \leq (\lambda - \Lambda_W)^{-1} I$. Combining this with (3.6), we have

$$\lambda \leq \frac{1}{\lambda - \Lambda_W} \frac{y^T T^T Ty}{y^T y} - \frac{y^T Wy}{y^T y} \leq \frac{\|T\|_2^2}{\lambda - \Lambda_W} + \lambda_W.$$

Solving this quadratic inequality for λ, we derive

$$\frac{\Lambda_W + \lambda_W - \sqrt{(\Lambda_W - \lambda_W)^2 + 4\|T\|_2^2}}{2} \leq \lambda \leq \frac{\Lambda_W + \lambda_W + \sqrt{(\Lambda_W - \lambda_W)^2 + 4\|T\|_2^2}}{2}. \tag{3.7}$$

It is direct to check that

$$\frac{\Lambda_W + \lambda_W - \sqrt{(\Lambda_W - \lambda_W)^2 + 4\|T\|_2^2}}{2} \leq \Lambda_W \leq \frac{\Lambda_W + \lambda_W + \sqrt{(\Lambda_W - \lambda_W)^2 + 4\|T\|_2^2}}{2},$$

which with $\lambda > \Lambda_W$ and (3.7) gives the desired upper bound in Lemma 3.3.

Next we consider the case that $\lambda < -\lambda_W$. Clearly $\lambda I - W \leq (\lambda + \lambda_W) I < 0$, which implies that $0 > (\lambda I - W)^{-1} \geq (\lambda + \lambda_W)^{-1} I$. Using this, we see from (3.6) that

$$\lambda \geq \frac{1}{\lambda + \lambda_W} \frac{y^T T^T Ty}{y^T y} - \frac{y^T Wy}{y^T y} \geq \frac{\|T\|_2^2}{\lambda + \lambda_W} - \Lambda_W,$$

solving this quadratic inequality, we get

$$\frac{-\lambda_W - \Lambda_W - \sqrt{(\lambda_W - \Lambda_W)^2 + 4\|T\|_2^2}}{2} \leq \lambda \leq \frac{-\lambda_W - \Lambda_W + \sqrt{(\lambda_W - \Lambda_W)^2 + 4\|T\|_2^2}}{2}.$$

This, along with the fact that

$$\frac{-\lambda_W - \Lambda_W - \sqrt{(\lambda_W - \Lambda_W)^2 + 4\|T\|_2^2}}{2} \le -\lambda_W \le \frac{-\lambda_W - \Lambda_W + \sqrt{(\lambda_W - \Lambda_W)^2 + 4\|T\|_2^2}}{2}$$

yields

$$\frac{-\lambda_W - \Lambda_W - \sqrt{(\lambda_W - \Lambda_W)^2 + 4\|T\|_2^2}}{2} \le \lambda \le -\lambda_W.$$

Summing up all the estimates above, we can conclude Lemma 3.2. □

By means of the spectral estimates in Lemma 3.2, we can now derive the spectral estimate of the matrix $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$. Noticing that

$$\begin{aligned}
\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T} &= \begin{pmatrix} \hat{L}^{-1} & 0 \\ 0 & \hat{L}^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \begin{pmatrix} \hat{L}^{-T} & 0 \\ 0 & \hat{L}^{-T} \end{pmatrix} \\
&= \begin{pmatrix} \hat{L}^{-1}A\hat{L}^{-T} & \hat{L}^{-1}B\hat{L}^{-T} \\ \hat{L}^{-1}B\hat{L}^{-T} & -\hat{L}^{-1}A\hat{L}^{-T} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & -\tilde{A} \end{pmatrix},
\end{aligned} \tag{3.8}$$

then a direct application of Lemma 3.2 leads to the following results.

Lemma 3.3. *Let $\mathcal{M}$ and $\mathcal{A}$ be defined as in* (1.2) *and* (3.2) *respectively, then the eigenvalues of the matrix $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$ lie in the range $[-\varrho, \varrho]$, where ϱ is given by*

$$\varrho = \frac{\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} + \sqrt{(\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + 4\|\tilde{B}\|_2^2}}{2}.$$

Remark 3.1. If we simply choose $\Lambda_{\tilde{A}} = \lambda_{\tilde{A}}$ for the bounds of the matrix $\tilde{A}$, then the spectral estimate of $\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}$ in Lemma 3.3 reduces to $\mathrm{sp}(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}) \subset [-\lambda_{\tilde{A}} - \|\tilde{B}\|_2, \lambda_{\tilde{A}} + \|\tilde{B}\|_2]$.

3.2. Spectral bounds of the inverse $(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}$

In this section we study how the eigenvalues of the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ approach zero. To do so, we estimate the spectral bounds of $(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}$. It is easy to verify that

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & -\tilde{A} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \tilde{B}\tilde{A}^{-1} & I \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & -\tilde{S} \end{pmatrix} \begin{pmatrix} I & \tilde{A}^{-1}\tilde{B} \\ 0 & I \end{pmatrix}.$$

This shows that

$$\begin{aligned}
(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1} &= \begin{pmatrix} I & -\tilde{A}^{-1}\tilde{B} \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{A}^{-1} & 0 \\ 0 & -\tilde{S}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\tilde{B}\tilde{A}^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} \tilde{A}^{-1} - \tilde{A}^{-1}\tilde{B}\tilde{S}^{-1}\tilde{B}\tilde{A}^{-1} & \tilde{A}^{-1}\tilde{B}\tilde{S}^{-1} \\ \tilde{S}^{-1}\tilde{B}\tilde{A}^{-1} & -\tilde{S}^{-1} \end{pmatrix}.
\end{aligned} \tag{3.9}$$

By Sherman–Morrison–Woodbury formula; see, e.g., [40], we have

$$\begin{aligned}
\tilde{A}^{-1} - \tilde{A}^{-1}\tilde{B}\tilde{S}^{-1}\tilde{B}\tilde{A}^{-1} &= \tilde{A}^{-1} - \tilde{A}^{-1}\tilde{B}(\tilde{A} + \tilde{B}\tilde{A}^{-1}\tilde{B})^{-1}\tilde{B}\tilde{A}^{-1} \\
&= \tilde{A}^{-1} - \tilde{A}^{-1}\tilde{B}(I + \tilde{A}^{-1}\tilde{B}\tilde{A}^{-1}\tilde{B})^{-1}\tilde{A}^{-1}\tilde{B}\tilde{A}^{-1} \\
&= (\tilde{A} + \tilde{B}\tilde{A}^{-1}\tilde{B})^{-1} = \tilde{S}^{-1}.
\end{aligned}$$

This together with (3.9) yields that

$$(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1} = \begin{pmatrix} \tilde{S}^{-1} & \tilde{A}^{-1}\tilde{B}\tilde{S}^{-1} \\ \tilde{S}^{-1}\tilde{B}\tilde{A}^{-1} & -\tilde{S}^{-1} \end{pmatrix}, \tag{3.10}$$

which has the same form as $\mathcal{R}$ in (3.4). Therefore, we can apply Lemma 3.2 for the upper and lower bounds of the eigenvalues of $(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}$, as stated in the following lemma.

Lemma 3.4. *Under the same setting and conditions as in Lemma 3.3, any eigenvalue μ of the matrix $(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}$ meets the following estimates:*

$$-\frac{\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4\nu}}{2} \le \mu \le \frac{\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4\nu}}{2}$$

where ν is the maximum eigenvalue of the matrix $\tilde{S}^{-1}(\tilde{B}\tilde{A}^{-2}\tilde{B})\tilde{S}^{-1}$.

3.3. Spectral estimate of the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$

We are now in a position to derive the bound of the eigenvalues of the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ by making use of the spectral estimates we have obtained in the previous two Sections 3.1 and 3.2.

Theorem 3.1. *Let $\mathcal{P}$ be defined as in* (3.1) *and ν the maximum eigenvalue of the matrix $\tilde{S}^{-1}(\tilde{B}\tilde{A}^{-2}\tilde{B})\tilde{S}^{-1}$, then the eigenvalues of the matrix $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ satisfy*

$$\text{sp}(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}) \subset [-a, -b] \cup [b, a],$$

where a and b are given by

$$a = \frac{\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} + \sqrt{(\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + 4\|\tilde{B}\|_2^2}}{2 + \eta^2 - \sqrt{4\eta^2 + \eta^4}},$$

$$b = \frac{4}{(\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4\nu})(2 + \eta^2 + \sqrt{4\eta^2 + \eta^4})}.$$

Proof. It follows from Lemma 3.1 that

$$\text{sp}(\mathcal{P}^{-1}\mathcal{A}\mathcal{A}^T\mathcal{P}^{-T}) \subset \left[\frac{2}{2 + \eta^2 + \sqrt{4\eta^2 + \eta^4}}, \frac{2}{2 + \eta^2 - \sqrt{4\eta^2 + \eta^4}}\right].$$

Then for any eigenvalue λ of the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$, we can derive by the spectral theorem and Lemmas 3.1–3.3 that

$$\lambda \leq \max_{z\neq 0} \frac{z^T\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}z}{z^Tz} = \max_{z\neq 0} \frac{z^T\mathcal{P}^{-1}\mathcal{A}\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}\mathcal{A}^T\mathcal{P}^{-T}z}{z^T\mathcal{P}^{-1}\mathcal{A}\mathcal{A}^T\mathcal{P}^{-T}z} \frac{z^T\mathcal{P}^{-1}\mathcal{A}\mathcal{A}^T\mathcal{P}^{-T}z}{z^Tz}$$

$$\leq \max_{z\neq 0} \frac{z^T\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}z}{z^Tz} \max_{z\neq 0} \frac{z^T\mathcal{P}^{-1}\mathcal{A}\mathcal{A}^T\mathcal{P}^{-T}z}{z^Tz} \leq \frac{\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} + \sqrt{(\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + 4\|\tilde{B}\|_2^2}}{2 + \eta^2 - \sqrt{4\eta^2 + \eta^4}},$$

and

$$\lambda \geq \min_{z\neq 0} \frac{z^T\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}z}{z^Tz} \geq \min_{z\neq 0} \frac{z^T\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T}z}{z^Tz} \max_{z\neq 0} \frac{z^T\mathcal{P}^{-1}\mathcal{A}\mathcal{A}^T\mathcal{P}^{-T}z}{z^Tz}$$

$$\geq -\frac{\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} + \sqrt{(\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + 4\|\tilde{B}\|_2^2}}{2 + \eta^2 - \sqrt{4\eta^2 + \eta^4}}.$$

On the other hand, it follows from Lemmas 3.1 and 3.4 that

$$\frac{1}{\lambda} \leq \max_{z\neq 0} \frac{z^T(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T})^{-1}z}{z^Tz} = \max_{z\neq 0} \frac{z^T\mathcal{P}^T\mathcal{M}^{-1}\mathcal{P}z}{z^Tz}$$

$$= \max_{z\neq 0} \frac{z^T\mathcal{P}^T\mathcal{A}^{-T}(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}\mathcal{A}^{-1}\mathcal{P}z}{z^T\mathcal{P}^T\mathcal{A}^{-T}\mathcal{A}^{-1}\mathcal{P}z} \frac{z^T\mathcal{P}^T\mathcal{A}^{-T}\mathcal{A}^{-1}\mathcal{P}z}{z^Tz}$$

$$\leq \max_{z\neq 0} \frac{z^T(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}z}{z^Tz} \max_{z\neq 0} \frac{z^T\mathcal{P}^T\mathcal{A}^{-T}\mathcal{A}^{-1}\mathcal{P}z}{z^Tz}$$

$$\leq \frac{(\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4\nu})(2 + \eta^2 + \sqrt{4\eta^2 + \eta^4})}{4},$$

and

$$\frac{1}{\lambda} \geq \min_{z\neq 0} \frac{z^T(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T})^{-1}z}{z^Tz} \geq \min_{z\neq 0} \frac{z^T(\mathcal{A}^{-1}\mathcal{M}\mathcal{A}^{-T})^{-1}z}{z^Tz} \max_{z\neq 0} \frac{z^T\mathcal{G}z}{z^Tz}$$

$$\geq -\frac{(\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4\nu})(2 + \eta^2 + \sqrt{4\eta^2 + \eta^4})}{4}.$$

Then for $\lambda > 0$, we deduce

$$\lambda \geq \frac{4}{(\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4\nu})(2 + \eta^2 + \sqrt{4\eta^2 + \eta^4})},$$

while for $\lambda < 0$, we have

$$\lambda \le -\frac{4}{(\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4v})(2 + \eta^2 + \sqrt{4\eta^2 + \eta^4})}.$$

This completes the proof of Theorem 3.1. □

Remark 3.2. The block triangular preconditioner $\mathcal{P}$ in (3.1) reduces to the block diagonal preconditioner for $\eta = 0$. In this case, a and b in Theorem 3.1 can be simplified to

$$a = \frac{\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} + \sqrt{(\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + 4\|\tilde{B}\|_2^2}}{2}, \quad b = \frac{2}{\Lambda_{\tilde{S}^{-1}} + \lambda_{\tilde{S}^{-1}} + \sqrt{(\Lambda_{\tilde{S}^{-1}} - \lambda_{\tilde{S}^{-1}})^2 + 4v}}.$$

Furthermore, if we choose $\Lambda_{\tilde{A}} = \lambda_{\tilde{A}}$ and $\Lambda_{\tilde{S}^{-1}} = \lambda_{\tilde{S}^{-1}}$, then $a = \Lambda_{\tilde{A}} + \|\tilde{B}\|_2$, $b = 1/(\Lambda_{\tilde{S}^{-1}} + \sqrt{v})$, and the estimate of Theorem 3.1 reduces to

$$\mathrm{sp}(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}) \subset \left[-\Lambda_{\tilde{A}} - \|\tilde{B}\|_2, -\frac{1}{\Lambda_{\tilde{S}^{-1}} + \sqrt{v}} \right] \cup \left[\frac{1}{\Lambda_{\tilde{S}^{-1}} + \sqrt{v}}, \Lambda_{\tilde{A}} + \|\tilde{B}\|_2 \right].$$

4. Block triangular preconditioner for one-sided preconditioning

Although the generalized saddle-point system (1.2) of our interest is symmetric, one may also like to apply some iteration methods for nonsymmetric systems in applications, see, e.g., [17–23]. To this end, we shall study the following block triangular preconditioner

$$\tilde{\mathcal{P}} = \begin{pmatrix} \hat{A} & 0 \\ B & \hat{A} \end{pmatrix} \tag{4.1}$$

to precondition the system (1.2) only from one side and analyze the spectrum of the preconditioned system. Then we will solve the following system instead of the original one (1.2):

$$\tilde{\mathcal{P}}^{-1}\mathcal{M}z = \tilde{\mathcal{P}}^{-1}b \quad \text{or} \quad \mathcal{M}\tilde{\mathcal{P}}^{-1}\hat{z} = b$$

with $\hat{z} = \tilde{\mathcal{P}}z$. Noting that the preconditioned systems $\tilde{\mathcal{P}}^{-1}\mathcal{M}$ and $\mathcal{M}\tilde{\mathcal{P}}^{-1}$ have the same eigenvalues, so we shall only analyze the spectral property of $\tilde{\mathcal{P}}^{-1}\mathcal{M}$. As the preconditioned system $\tilde{\mathcal{P}}^{-1}\mathcal{M}$ is nonsymmetric, we need to discuss its real and complex eigenvalues respectively. For both real and complex cases, we shall study the eigen-system of a matrix $\mathcal{W}$ that is similar to $\tilde{\mathcal{P}}^{-1}\mathcal{M}$. For the real eigenvalues, we will do the same as we did in the proof of Lemma 3.3 by considering a one-variable quadratic inequality arising from the eigen-system. But the deriving process of the inequality is much more complicated than before. And then we achieve the estimates by solving the resulting inequality. The study of the complex eigenvalues is much more tricky and delicate. By separating the real and imaginary parts of the quadratic form of the resultant equation, we achieve the desired estimates by using the important relation between the real and imaginary parts. In the remainder of this section, we follow this general guideline to investigate the spectral properties of $\tilde{\mathcal{P}}^{-1}\mathcal{M}$. First we can easily verify that

$$\begin{aligned}
\tilde{\mathcal{P}}^{-1}\mathcal{M} &= \begin{pmatrix} \hat{A}^{-1} & 0 \\ -\hat{A}^{-1}B\hat{A}^{-1} & \hat{A}^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \\
&= \begin{pmatrix} \hat{A}^{-1}A & \hat{A}^{-1}B \\ -\hat{A}^{-1}B\hat{A}^{-1}A + \hat{A}^{-1}B & -\hat{A}^{-1}B\hat{A}^{-1}B - \hat{A}^{-1}A \end{pmatrix},
\end{aligned}$$

which is similar to

$$\begin{aligned}
\mathcal{W} &=: \begin{pmatrix} \hat{A}^{-\frac{1}{2}}A\hat{A}^{-\frac{1}{2}} & \hat{A}^{-\frac{1}{2}}B\hat{A}^{-\frac{1}{2}} \\ -\hat{A}^{-\frac{1}{2}}B\hat{A}^{-1}A\hat{A}^{-\frac{1}{2}} + \hat{A}^{-\frac{1}{2}}B\hat{A}^{-\frac{1}{2}} & -\hat{A}^{-\frac{1}{2}}B\hat{A}^{-1}B\hat{A}^{-\frac{1}{2}} - \hat{A}^{-\frac{1}{2}}A\hat{A}^{-\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} \tilde{A} & \tilde{B} \\ -\tilde{B}\tilde{A} + \tilde{B} & -\tilde{B}^2 - \tilde{A} \end{pmatrix}. \tag{4.2}
\end{aligned}$$

Thereby, the preconditioned matrix $\tilde{\mathcal{P}}^{-1}\mathcal{M}$ and $\mathcal{W}$ have the same eigenvalues. So it suffices for us to analyze the eigenvalues of the matrix $\mathcal{W}$. Let λ be an eigenvalue of $\mathcal{W}$ and $(x^T, y^T)^T$ be the corresponding eigenvector. Then we have

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ -\tilde{B}\tilde{A} + \tilde{B} & -\tilde{B}^2 - \tilde{A} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

equivalently,

$$\begin{cases} \tilde{A}x + \tilde{B}y = \lambda x, \\ \tilde{B}\tilde{A}x - \tilde{B}x + \tilde{B}^2 y + \tilde{A}y = -\lambda y. \end{cases} \tag{4.3}$$

Substituting the first equation of (4.3) into the second one leads to

$$\begin{cases} \tilde{A}x + \tilde{B}y = \lambda x, \\ (\lambda - 1)\tilde{B}x + \tilde{A}y = -\lambda y. \end{cases} \tag{4.4}$$

Using this eigen-system, we can now derive the spectral estimates of the matrix $\mathcal{W}$ in (4.2), first for real eigenvalues in Theorem 4.1, then for complex eigenvalues in Theorem 4.2. For this purpose, we introduce two notations:

$$\varsigma = -\frac{\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2 + \sqrt{(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)}}{2},$$

$$\Phi = \frac{\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2 + \sqrt{(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)}}{2}.$$

Theorem 4.1. *Let $\mathcal{W}$ be the matrix defined as in* (4.2)*, then all the real eigenvalues λ of $\mathcal{W}$ meet the following estimates:*
  1. *if $\Lambda_{\tilde{A}} \geq 1$, then $\lambda \in [\varsigma, \max\{\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}\}]$;*
  2. *if $\Lambda_{\tilde{A}} < 1$ and $\lambda_{\tilde{A}} < 1$, then $\lambda \in [\varsigma, \Phi] \cup \{1\}$.*

Proof. Let $(x^T, y^T)^T$ be the corresponding eigenvector of $\mathcal{W}$. If $y = 0$, then we get from (4.4) that $\tilde{A}x = \lambda x$ and $(\lambda - 1)\tilde{B}x = 0$, which imply that $\lambda \in [-\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$ or $\lambda = 1$. This also suggests us to consider $\lambda \notin [-\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$ if $y \neq 0$. Clearly $\lambda I - \tilde{A}$ is nonsingular in this case, and it follows from the first equality of (4.4) that $x = (\lambda I - \tilde{A})^{-1}\tilde{B}y$. This together with (4.4) yields that

$$(\lambda - 1)\tilde{B}(\lambda I - \tilde{A})^{-1}\tilde{B}y + \tilde{A}y = -\lambda y,$$

by multiplying the above equality from the left by y^T/y^Ty, we obtain

$$-\lambda = (\lambda - 1)\frac{y^T\tilde{B}(\lambda I - \tilde{A})^{-1}\tilde{B}y}{y^Ty} + \frac{y^T\tilde{A}y}{y^Ty}. \tag{4.5}$$

Next we estimate the eigenvalue λ in two different cases.

We first consider the case that $\lambda > \Lambda_{\tilde{A}}$. Clearly $\lambda I - \tilde{A} \geq (\lambda - \Lambda_{\tilde{A}})I > 0$, which implies that $0 < (\lambda I - \tilde{A})^{-1} \leq (\lambda - \Lambda_{\tilde{A}})^{-1}I$. If $\Lambda_{\tilde{A}} \geq 1$, then $\lambda > 1$, yielding that

$$-\lambda \geq \frac{y^T\tilde{A}y}{y^Ty} \geq -\lambda_{\tilde{A}},$$

i.e., $\lambda \leq \lambda_{\tilde{A}}$. Combining this with the result $\lambda \in [-\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$ for $y = 0$, we see $\lambda \leq \max\{\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}\}$.

If $\Lambda_{\tilde{A}} < 1$ and $\lambda_{\tilde{A}} < 1$, we can assert that $\lambda < 1$. Otherwise, in the same manner as we did earlier, we can deduce that $1 \leq \lambda \leq \lambda_{\tilde{A}}$, which contradicts the fact that $\lambda_{\tilde{A}} < 1$. Then combining with (4.5), we have

$$-\lambda \geq \frac{\lambda - 1}{\lambda - \Lambda_{\tilde{A}}}\frac{y^T\tilde{B}^2y}{y^Ty} + \frac{y^T\tilde{A}y}{y^Ty} \geq \frac{\lambda - 1}{\lambda - \Lambda_{\tilde{A}}}\|\tilde{B}\|_2^2 - \lambda_{\tilde{A}}.$$

By simple calculations, we derive

$$\lambda^2 - (\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)\lambda - \|\tilde{B}\|_2^2 + \lambda_{\tilde{A}}\Lambda_{\tilde{A}} \leq 0.$$

This, along with the simple estimate that

$$\begin{aligned}
&(\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2) \\
&= (\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + \|\tilde{B}\|_2^4 - 2\|\tilde{B}\|_2^2\Lambda_{\tilde{A}} - 2\|\tilde{B}\|_2^2\lambda_{\tilde{A}} + 4\|\tilde{B}\|_2^2 \\
&> (\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + \|\tilde{B}\|_2^4 - 2\|\tilde{B}\|_2^2\Lambda_{\tilde{A}} - 2\|\tilde{B}\|_2^2\lambda_{\tilde{A}} + 4\|\tilde{B}\|_2^2\Lambda_{\tilde{A}} \\
&= (\Lambda_{\tilde{A}} - \lambda_{\tilde{A}})^2 + \|\tilde{B}\|_2^4 + 2\|\tilde{B}\|_2^2(\Lambda_{\tilde{A}} - \lambda_{\tilde{A}}) \\
&= (\Lambda_{\tilde{A}} - \lambda_{\tilde{A}} + \|\tilde{B}\|_2^2)^2 \geq 0,
\end{aligned}$$

leads to

$$\frac{\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} - \|\tilde{B}\|_2^2 - \sqrt{(\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)}}{2} \leq \lambda \leq \Phi. \tag{4.6}$$

Furthermore, we can directly check the following relations

$$\frac{\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} - \|\tilde{B}\|_2^2 - \sqrt{(\Lambda_{\tilde{A}} + \lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)}}{2} < \Lambda_{\tilde{A}} < \Phi < 1$$

hold for $\Lambda_{\tilde{A}} < 1$ and $\lambda_{\tilde{A}} < 1$. This, with $\Lambda_{\tilde{A}} < \lambda < 1$ and (4.6), gives $\Lambda_{\tilde{A}} < \lambda \leq \Phi$.

Next we consider the case that $\lambda < -\lambda_{\tilde{A}}$. Clearly $\tilde{A} - \lambda I \geq (-\lambda_{\tilde{A}} - \lambda)I > 0$, which yields that $0 < (\tilde{A} - \lambda I)^{-1} \leq (-\lambda_{\tilde{A}} - \lambda)^{-1}I$. Using this, we directly see from (4.5) that

$$- \lambda = (1 - \lambda)\frac{y^T\tilde{B}(\tilde{A} - \lambda I)^{-1}\tilde{B}y}{y^Ty} + \frac{y^T\tilde{A}y}{y^Ty} \leq \frac{1 - \lambda}{-\lambda_{\tilde{A}} - \lambda}\frac{y^T\tilde{B}^2y}{y^Ty} + \frac{y^T\tilde{A}y}{y^Ty} \leq \frac{1 - \lambda}{-\lambda_{\tilde{A}} - \lambda}\|\tilde{B}\|_2^2 + \Lambda_{\tilde{A}},$$

by a further simplification, we obtain

$$\lambda^2 + (\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2)\lambda + \lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2 \leq 0.$$

Combining this with the following fact that

$$(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2) = (\lambda_{\tilde{A}} - \Lambda_{\tilde{A}})^2 + \|\tilde{B}\|_2^4 + 2\|\tilde{B}\|_2^2(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + 2) \geq 0,$$

we deduce

$$\varsigma \leq \lambda \leq \frac{-(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2) + \sqrt{(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)}}{2}.$$

This, along with the fact that

$$\varsigma \leq -\lambda_{\tilde{A}} \leq \frac{-(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2) + \sqrt{(\lambda_{\tilde{A}} + \Lambda_{\tilde{A}} + \|\tilde{B}\|_2^2)^2 - 4(\lambda_{\tilde{A}}\Lambda_{\tilde{A}} - \|\tilde{B}\|_2^2)}}{2},$$

leads to the estimate that $\varsigma \leq \lambda < -\lambda_{\tilde{A}}$. This completes the proof of Theorem 4.1. □

The next theorem gives some bounds on the complex eigenvalues of matrix $\mathcal{W}$ in (4.2).

Theorem 4.2. Let $\mathcal{W}$ be defined as in (4.2). If $\mathcal{W}$ has a complex eigenvalue λ with $\text{Im}(\lambda) \neq 0$, then $\Lambda_{\tilde{A}} \geq 1$. Moreover, if $\lambda_{\tilde{A}} \geq 1$, λ can be bounded as follows:

$$|\text{Im}(\lambda)| \leq \sqrt{\Lambda_{\tilde{A}} - 1}\|\tilde{B}\|_2,$$

$$1 - \sqrt{(\Lambda_{\tilde{A}} - 1)(\lambda_{\tilde{A}} - 1)} \leq \text{Re}(\lambda) \leq 1 + \sqrt{(\Lambda_{\tilde{A}} - 1)(\lambda_{\tilde{A}} - 1)}.$$

Proof. Let $\lambda = a + bi$, with $a \in \mathbb{R}$, $0 \neq b \in \mathbb{R}$ and $i = \sqrt{-1}$, be a complex eigenvalue of $\mathcal{W}$, and $z = (x^T, y^T)^T$ be an eigenvector corresponding to λ. It is easy to see that $\lambda I - \tilde{A}$ is nonsingular. Then similarly to what we did in the proof of Theorem 4.1, we have

$$- \lambda y^*y = (\lambda - 1)y^*\tilde{B}(\lambda I - \tilde{A})^{-1}\tilde{B}y + y^*\tilde{A}y. \tag{4.7}$$

Let $\tilde{A} = UDU^*$ be the eigen-decomposition of $\tilde{A}$, where $U \in \mathbb{C}^{n \times n}$ is an unitary matrix and $D = \text{diag}(\rho_1, \rho_2, \ldots, \rho_n)$ with $\rho_1 = -\lambda_{\tilde{A}}$ and $\rho_n = \Lambda_{\tilde{A}}$. Using this eigen-decomposition, we derive from (4.7) that

$$- \lambda y^*y = (\lambda - 1)y^*\tilde{B}U(\lambda I - D)^{-1}U^*\tilde{B}y + y^*\tilde{A}y. \tag{4.8}$$

It is easy to see for any $1 \leq j \leq n$ that

$$(\lambda - \rho_j)^{-1} = \frac{1}{a + bi - \rho_j} = \frac{a - \rho_j - bi}{(a - \rho_j)^2 + b^2},$$

which implies

$$(\lambda I - D)^{-1} = \text{diag}\left(\frac{a - \rho_1 - bi}{(a - \rho_1)^2 + b^2}, \ldots, \frac{a - \rho_n - bi}{(a - \rho_n)^2 + b^2}\right).$$

Then it follows from this equality and (4.8) that

$$- ay^*y = (a - 1)H_r + b^2H_i + y^*\tilde{A}y, \tag{4.9}$$

$$- y^*y = H_r - (a - 1)H_i, \tag{4.10}$$

where H_r and H_i are given by

$$H_r = y^*\tilde{B}U\text{diag}\left(\frac{a - \rho_1}{(a - \rho_1)^2 + b^2}, \ldots, \frac{a - \rho_n}{(a - \rho_n)^2 + b^2}\right)U^*\tilde{B}y, \tag{4.11}$$

$$H_i = y^*\tilde{B}U\text{diag}\left(\frac{1}{(a - \rho_1)^2 + b^2}, \ldots, \frac{1}{(a - \rho_n)^2 + b^2}\right)U^*\tilde{B}y. \tag{4.12}$$

We now claim that $y \neq 0$. If this is not true, we get from (4.4) that $\tilde{A}x = \lambda x$ and $\tilde{B}x = 0$, which contradicts the fact that $b \neq 0$. It follows from (4.11) and (4.12) that

$$(a - \Lambda_{\tilde{A}})H_i \leq H_r \leq (a + \lambda_{\tilde{A}})H_i. \tag{4.13}$$

This together with (4.10) gives

$$(a - 1)H_i - y^*y \geq (a - \Lambda_{\tilde{A}})H_i.$$

Hence we have $(\Lambda_{\tilde{A}} - 1)H_i \geq y^*y$, which shows $\Lambda_{\tilde{A}} \geq 1$, and

$$1 \leq (\Lambda_{\tilde{A}} - 1)\frac{H_i}{y^*y} \leq (\Lambda_{\tilde{A}} - 1)\frac{\|\tilde{B}\|_2^2}{b^2},$$

or

$$|b| \leq \sqrt{\Lambda_{\tilde{A}} - 1}\|\tilde{B}\|_2.$$

On the other hand, combining (4.9) and (4.10), we derive

$$H_r = \frac{(a - a^2 - b^2)y^*y - (a - 1)y^*\tilde{A}y}{(a - 1)^2 + b^2}, \quad H_i = \frac{-y^*y - y^*\tilde{A}y}{(a - 1)^2 + b^2}.$$

Then we obtain using (4.13) that

$$(a - \Lambda_{\tilde{A}})\frac{-y^*y - y^*\tilde{A}y}{(a - 1)^2 + b^2} \leq \frac{(a - a^2 - b^2)y^*y - (a - 1)y^*\tilde{A}y}{(a - 1)^2 + b^2},$$

which implies that

$$(a - \Lambda_{\tilde{A}})(y^*y + y^*\tilde{A}y) \geq (a^2 + b^2 - a)y^*y + (a - 1)y^*\tilde{A}y.$$

This together with the fact that $\mathrm{sp}(\tilde{A}) \subset [-\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$ yields

$$0 \geq a^2 + b^2 - 2a + \Lambda_{\tilde{A}} + (\Lambda_{\tilde{A}} - 1)\frac{y^*\tilde{A}y}{y^*y} \geq a^2 - 2a + \Lambda_{\tilde{A}} - \lambda_{\tilde{A}}(\Lambda_{\tilde{A}} - 1),$$

hence it is easy to see that

$$1 - \sqrt{(\Lambda_{\tilde{A}} - 1)(\lambda_{\tilde{A}} - 1)} \leq a \leq 1 + \sqrt{(\Lambda_{\tilde{A}} - 1)(\lambda_{\tilde{A}} - 1)},$$

and completes the proof of Theorem 4.2. □

Remark 4.1. We see from Theorem 4.2 that all the eigenvalues of $\mathcal{W}$ are real if $\Lambda_{\tilde{A}} < 1$. On the other hand, we remark that the case with $\lambda_{\tilde{A}} < 1$ was not addressed in Theorem 4.2. But this can be easily classified as the case with $\lambda_{\tilde{A}} \geq 1$, by assuming $\mathrm{sp}(\tilde{A}) \subset [-\Lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$ instead of $\mathrm{sp}(\tilde{A}) \subset [-\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$.

As we recall at the beginning of Section 4, the preconditioned matrix $\tilde{\mathcal{P}}^{-1}\mathcal{M}$ is similar to $\mathcal{W}$, hence both have the same eigenvalues. Then the following results are direct consequences of Theorems 4.1 and 4.2.

Theorem 4.3. *Let $\tilde{\mathcal{P}}$ be defined as in (4.1), then the following estimates hold for the eigenvalues λ of $\tilde{\mathcal{P}}^{-1}\mathcal{M}$:*
(1) *if $|\mathrm{Im}(\lambda)| = 0$, then*

$$\lambda \in \begin{cases} [\varsigma, \max\{\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}\}], & \text{if } \Lambda_{\tilde{A}} \geq 1, \\ [\varsigma, \Phi] \cup \{1\}, & \text{if } \Lambda_{\tilde{A}} < 1 \text{ and } \lambda_{\tilde{A}} < 1. \end{cases}$$

(2) *if $|\mathrm{Im}(\lambda)| \neq 0$, then $\Lambda_{\tilde{A}} \geq 1$. Moreover, if $\lambda_{\tilde{A}} \geq 1$, then*

$$1 - \sqrt{(\Lambda_{\tilde{A}} - 1)(\lambda_{\tilde{A}} - 1)} \leq \mathrm{Re}(\lambda) \leq 1 + \sqrt{(\Lambda_{\tilde{A}} - 1)(\lambda_{\tilde{A}} - 1)}, \quad |\mathrm{Im}(\lambda)| \leq \sqrt{\Lambda_{\tilde{A}} - 1}\|\tilde{B}\|_2.$$

Remark 4.2. If $\Lambda_{\tilde{A}}$ and $\lambda_{\tilde{A}}$ are close to 1, then we can easily see from Theorem 4.3 that the real parts of all the complex eigenvalues with $|\mathrm{Im}(\lambda)| \neq 0$ of the preconditioned system $\tilde{\mathcal{P}}^{-1}\mathcal{M}$ are positive and clustered around 1 while the imaginary parts are small.

5. Spectral estimates of the preconditioned system by diagonal preconditioner $\mathcal{P}_s$

The diagonal preconditioner $\mathcal{P}_s$ in (1.3) was proposed in [1] for solving the generalized saddle-point system (1.2) when it arises from the edge element discretization of the PML system of an electromagnetic wave scattering problem. The preconditioned system $\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}}$ was investigated in a general framework in [1] under some abstract assumptions,

resulting in only some quantitative spectral estimates. And it appears to be still rather difficult to verify all the abstract assumptions in [1]. In this section, we shall derive some more explicit bounds for the positive and negative eigenvalues of the preconditioned system $\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}$, following a purely algebraic argument. For this purpose, we introduce the following few helpful matrices

$$\tilde{A} = \hat{A}^{-\frac{1}{2}} A \hat{A}^{-\frac{1}{2}}, \quad \tilde{H} = \hat{A}^{-\frac{1}{2}} B \hat{S}^{-\frac{1}{2}}, \quad \tilde{A}_s = \hat{S}^{-\frac{1}{2}} A \hat{S}^{-\frac{1}{2}}, \quad \tilde{S} = \hat{S}^{-\frac{1}{2}} S \hat{S}^{-\frac{1}{2}}.$$

We are now ready to estimate the spectrum of the preconditioned matrix $\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}$. For this purpose, we first write

$$
\begin{aligned}
\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}} &= \begin{pmatrix} \hat{A}^{-\frac{1}{2}} & 0 \\ 0 & \hat{S}^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \begin{pmatrix} \hat{A}^{-\frac{1}{2}} & 0 \\ 0 & \hat{S}^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{H} \\ \tilde{H}^T & -\tilde{A}_s \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ \tilde{H}^T \tilde{A}^{-1} & I \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & -\hat{S}^{-\frac{1}{2}} S \hat{S}^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & \tilde{A}^{-1} \tilde{H} \\ 0 & I \end{pmatrix} := \mathcal{L} \mathcal{D} \mathcal{L}^T.
\end{aligned}
\tag{5.1}
$$

Next, we shall make use of this factorization to estimate respectively the lower and upper bounds of all positive and negative eigenvalues of $\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}$. First, for any positive eigenvalue λ we can directly see that

$$
\begin{aligned}
\lambda &\leq \lambda_{\max}\left(\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}\right) = \max_{z \neq 0} \frac{z^T \mathcal{L} \mathcal{D} \mathcal{L}^T z}{z^T z} = \max_{z \neq 0} \frac{z^T \mathcal{L} \mathcal{D} \mathcal{L}^T z}{z^T \mathcal{L} \mathcal{L}^T z} \frac{z^T \mathcal{L} \mathcal{L}^T z}{z^T z} \\
&\leq \max_{z \neq 0} \frac{z^T \mathcal{D} z}{z^T z} \max_{z \neq 0} \frac{z^T \mathcal{L} \mathcal{L}^T z}{z^T z},
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
\frac{1}{\lambda} &\leq \lambda_{\max}\left(\left(\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}\right)^{-1}\right) = \max_{z \neq 0} \frac{z^T \mathcal{L}^{-T} \mathcal{D}^{-1} \mathcal{L}^{-1} z}{z^T z} \\
&= \max_{z \neq 0} \frac{z^T \mathcal{L}^{-T} \mathcal{D}^{-1} \mathcal{L}^{-1} z}{z^T \mathcal{L}^{-T} \mathcal{L}^{-1} z} \frac{z^T (\mathcal{L} \mathcal{L}^T)^{-1} z}{z^T z} \leq \max_{z \neq 0} \frac{z^T \mathcal{D}^{-1} z}{z^T z} \max_{z \neq 0} \frac{z^T (\mathcal{L} \mathcal{L}^T)^{-1} z}{z^T z}.
\end{aligned}
\tag{5.3}
$$

Then it follows from the definition of $\mathcal{L}$ that

$$\mathcal{L} \mathcal{L}^T = \begin{pmatrix} I & 0 \\ \tilde{H}^T \tilde{A}^{-1} & I \end{pmatrix} \begin{pmatrix} I & \tilde{A}^{-1} \tilde{H} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & \tilde{A}^{-1} \tilde{H} \\ \tilde{H}^T \tilde{A}^{-1} & I + \tilde{H}^T \tilde{A}^{-2} \tilde{H} \end{pmatrix}.$$

Consider the singular value decomposition

$$\tilde{A}^{-1} \tilde{H} = V \Sigma U^T,$$

where $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ with $\sigma_i \geq 0$ for all $i = 1, 2, \ldots, n$, U and V are two orthonormal $n \times n$ matrices. Then we can easily see

$$\mathcal{L} \mathcal{L}^T = \begin{pmatrix} I & V \Sigma U^T \\ U \Sigma V^T & I + U \Sigma^2 U^T \end{pmatrix},$$

which is similar to the following simple matrix

$$\mathcal{D} = \begin{pmatrix} I & \Sigma \\ \Sigma & I + \Sigma^2 \end{pmatrix}.$$

Then it is not difficult to deduce that

$$\text{sp}(\mathcal{L} \mathcal{L}^T) \subset \left[\frac{2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2}}{2}, \frac{2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2}}{2} \right],
\tag{5.4}$$

where ℓ_1 is the maximum eigenvalues of $\tilde{H}^T \tilde{A}^{-2} \tilde{H}$. Combining this with (5.2) and (5.3) yields

$$\frac{\min\{\Gamma_{\tilde{A}}, \gamma_{\tilde{S}}\}(2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2})}{2} \leq \lambda \leq \frac{\max\{\lambda_{\tilde{S}}, \Lambda_{\tilde{A}}\}(2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2})}{2}.
\tag{5.5}$$

Now for any negative eigenvalue λ of $\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}$, it follows from (5.4) that

$$
\begin{aligned}
\lambda &\geq \lambda_{\min}\left(\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}\right) \geq \min_{z \neq 0} \frac{z^T \mathcal{D} z}{z^T z} \max_{z \neq 0} \frac{z^T \mathcal{L} \mathcal{L}^T z}{z^T z} \\
&= -\frac{\max\{\lambda_{\tilde{A}}, \Lambda_{\tilde{S}}\}(2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2})}{2},
\end{aligned}
$$

and

$$\frac{1}{\lambda} \geq \lambda_{\min}\left(\left(\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}}\right)^{-1}\right) \geq \min_{z \neq 0}\frac{z^T\mathcal{D}^{-1}z}{z^Tz}\max_{z \neq 0}\frac{z^T(\mathcal{L}\mathcal{L}^T)^{-1}z}{z^Tz}$$

$$= \frac{-2}{\min\{\Gamma_{\tilde{S}}, \gamma_{\tilde{A}}\}(2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2})}.$$

This shows that

$$-\frac{\max\{\lambda_{\tilde{A}}, \Lambda_{\tilde{S}}\}(2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2})}{2} \leq \lambda \leq -\frac{\min\{\Gamma_{\tilde{S}}, \gamma_{\tilde{A}}\}(2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2})}{2}. \tag{5.6}$$

Combining (5.5) and (5.6), we can now conclude the following results.

Theorem 5.1. *Let $\mathcal{M}$ and $\mathcal{P}_s$ be defined as in (1.2) and (1.3), ℓ_1 be the maximum eigenvalues of $\tilde{H}^T\tilde{A}^{-2}\tilde{H}$, then all the eigenvalues of the preconditioned matrix $\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}}$ lie in the range $\mathcal{I}^- \cup \mathcal{I}^+$, with*

$$\mathcal{I}^+ = \left[\frac{\min\{\Gamma_{\tilde{A}}, \gamma_{\tilde{S}}\}(2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2})}{2}, \frac{\max\{\lambda_{\tilde{S}}, \Lambda_{\tilde{A}}\}(2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2})}{2}\right] \subset \mathbb{R}^+,$$

$$\mathcal{I}^- = \left[-\frac{\max\{\lambda_{\tilde{A}}, \Lambda_{\tilde{S}}\}(2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2})}{2}, -\frac{\min\{\Gamma_{\tilde{S}}, \gamma_{\tilde{A}}\}(2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2})}{2}\right] \subset \mathbb{R}^-.$$

As can be seen from the above theorem that the spectral bounds of $\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}}$ can be made independent of the spectral bounds of $\tilde{A}_s$ and $\tilde{H}^T\tilde{H}$. On the other hand, we can also make use of the spectral bounds of $\tilde{A}_s$ and $\tilde{H}^T\tilde{H}$, then we can derive an alternative estimate of the spectrum of the preconditioned system $\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}}$, as it is done in the following lemma.

Lemma 5.1. *Let $\mathcal{M}$ and $\mathcal{P}_s$ be defined as in (1.2) and (1.3), ℓ_2 be the maximum eigenvalue of $\tilde{H}^T\tilde{H}$, then all the eigenvalues λ of the matrix $\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}}$ lie in the following range*

$$\frac{-\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s} - \sqrt{(\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s})^2 + 4\ell_2}}{2} \leq \lambda \leq \frac{\lambda_{\tilde{A}_s} + \Lambda_{\tilde{A}} + \sqrt{(\lambda_{\tilde{A}_s} - \Lambda_{\tilde{A}})^2 + 4\ell_2}}{2}.$$

Proof. Let λ and $(x^T, y^T)^T$ be the eigenvalue and eigenvector of the preconditioned matrix $\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}}$, then using

$$\mathcal{P}_s^{-\frac{1}{2}}\mathcal{M}\mathcal{P}_s^{-\frac{1}{2}} = \begin{pmatrix} \hat{A}^{-\frac{1}{2}} & 0 \\ 0 & \hat{S}^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\begin{pmatrix} \hat{A}^{-\frac{1}{2}} & 0 \\ 0 & \hat{S}^{-\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{A}^{-\frac{1}{2}}A\hat{A}^{-\frac{1}{2}} & \hat{A}^{-\frac{1}{2}}B\hat{S}^{-\frac{1}{2}} \\ \hat{S}^{-\frac{1}{2}}B\hat{A}^{-\frac{1}{2}} & -\hat{S}^{-\frac{1}{2}}A\hat{S}^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{H} \\ \tilde{H}^T & -\tilde{A}_s \end{pmatrix}, \tag{5.7}$$

it is easy to see

$$\tilde{A}x + \tilde{H}y = \lambda x, \quad \tilde{H}^Tx - \tilde{A}_sy = \lambda y. \tag{5.8}$$

If $y = 0$, we see readily $\tilde{H}^Tx = 0$ and $\tilde{A}x = \lambda x$, so we know $\lambda \in [-\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$. This suggests us to consider only the case that $\lambda \notin [-\lambda_{\tilde{A}}, \Lambda_{\tilde{A}}]$ if $y \neq 0$. Clearly $\lambda I - \tilde{A}$ is nonsingular in this case, and it follows from the first equality of (5.8) that $x = (\lambda I - \tilde{A})^{-1}\tilde{H}y$. Substituting it into the second equality in (5.8) yields

$$\tilde{H}^T(\lambda I - \tilde{A})^{-1}\tilde{H}y - \tilde{A}_sy = \lambda y,$$

then multiplying the above equality from the left by y^T/y^Ty, we obtain

$$\lambda = \frac{y^T\tilde{H}^T(\lambda I - \tilde{A})^{-1}\tilde{H}y}{y^Ty} - \frac{y^T\tilde{A}_sy}{y^Ty}. \tag{5.9}$$

Next we analyze in two cases. We first consider the case that $\lambda > \Lambda_{\tilde{A}}$. Clearly $\lambda I - \tilde{A} \geq (\lambda - \Lambda_{\tilde{A}})I > 0$, so it holds that $0 < (\lambda I - \tilde{A})^{-1} \leq (\lambda - \Lambda_{\tilde{A}})^{-1}I$. Combining this with (5.9) gives

$$\lambda \leq \frac{1}{\lambda - \Lambda_{\tilde{A}}}\frac{y^T\tilde{H}^T\tilde{H}y}{y^Ty} - \frac{y^T\tilde{A}_sy}{y^Ty} \leq \frac{\ell_2}{\lambda - \Lambda_{\tilde{A}}} + \lambda_{\tilde{A}_s},$$

solving the above quadratic inequality for λ, we obtain

$$\frac{\lambda_{\tilde{A}_s} + \Lambda_{\tilde{A}} - \sqrt{(\lambda_{\tilde{A}_s} - \Lambda_{\tilde{A}})^2 + 4\ell_2}}{2} \leq \lambda \leq \frac{\lambda_{\tilde{A}_s} + \Lambda_{\tilde{A}} + \sqrt{(\lambda_{\tilde{A}_s} - \Lambda_{\tilde{A}})^2 + 4\ell_2}}{2}.$$

On the other hand, we can directly check that

$$\frac{\lambda_{\tilde{A}_s} + \Lambda_{\tilde{A}} - \sqrt{(\lambda_{\tilde{A}_s} - \Lambda_{\tilde{A}})^2 + 4\ell_2}}{2} < \Lambda_{\tilde{A}} < \frac{\lambda_{\tilde{A}_s} + \Lambda_{\tilde{A}} + \sqrt{(\lambda_{\tilde{A}_s} - \Lambda_{\tilde{A}})^2 + 4\ell_2}}{2}$$

this with $\lambda > \Lambda_{\tilde{A}}$ gives

$$\Lambda_{\tilde{A}} < \lambda \leq \frac{\lambda_{\tilde{A}_s} + \Lambda_{\tilde{A}} + \sqrt{(\lambda_{\tilde{A}_s} - \Lambda_{\tilde{A}})^2 + 4\ell_2}}{2}. \tag{5.10}$$

Next we consider the case that $\lambda < -\lambda_{\tilde{A}}$. Clearly $\lambda I - \tilde{A} \leq (\lambda + \lambda_{\tilde{A}})I < 0$, so we have $0 > (\lambda I - \tilde{A})^{-1} \geq (\lambda + \lambda_{\tilde{A}})^{-1}I$. Then it follows from (5.9) that

$$\lambda \geq \frac{1}{\lambda + \lambda_{\tilde{A}}} \frac{y^T \tilde{H}^T \tilde{H} y}{y^T y} - \frac{y^T \tilde{A}_s y}{y^T y} \geq \frac{\ell_2}{\lambda + \lambda_{\tilde{A}}} - \Lambda_{\tilde{A}_s}.$$

By solving this inequality, we get the bounds that

$$\frac{-\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s} - \sqrt{(\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s})^2 + 4\ell_2}}{2} \leq \lambda \leq \frac{-\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s} + \sqrt{(\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s})^2 + 4\ell_2}}{2}.$$

This, along with the fact that

$$\frac{-\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s} - \sqrt{(\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s})^2 + 4\ell_2}}{2} < -\lambda_{\tilde{A}} < \frac{-\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s} + \sqrt{(\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s})^2 + 4\ell_2}}{2}$$

yields

$$\frac{-\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s} - \sqrt{(\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s})^2 + 4\ell_2}}{2} \leq \lambda < -\lambda_{\tilde{A}}. \tag{5.11}$$

Combining this with (5.10)–(5.11) and the case with $y = 0$, we complete the proof of Lemma 5.1. $\square$

To proceed our estimates, we introduce two more notations:

$$\mathcal{U} = \min\left\{ \frac{\max\{\lambda_{\tilde{S}}, \Lambda_{\tilde{A}}\}(2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2})}{2}, \frac{\lambda_{\tilde{A}_s} + \Lambda_{\tilde{A}} + \sqrt{(\lambda_{\tilde{A}_s} - \Lambda_{\tilde{A}})^2 + 4\ell_2}}{2} \right\},$$

$$\mathcal{V} = \min\left\{ \frac{\max\{\lambda_{\tilde{A}}, \Lambda_{\tilde{S}}\}(2 + \ell_1 + \sqrt{4\ell_1 + \ell_1^2})}{2}, \frac{\lambda_{\tilde{A}} + \Lambda_{\tilde{A}_s} + \sqrt{(\lambda_{\tilde{A}} - \Lambda_{\tilde{A}_s})^2 + 4\ell_2}}{2} \right\}.$$

Then we come to the following conclusion directly from Theorem 5.1 and Lemma 5.1.

Theorem 5.2. *Let $\mathcal{M}$ and $\mathcal{P}_s$ be defined as in (1.2) and (1.3), let ℓ_1 and ℓ_2 be the maximum eigenvalue of $\tilde{H}^T \tilde{A}^{-2} \tilde{H}$ and $\tilde{H}^T \tilde{H}$, respectively. Then all the eigenvalues λ of the preconditioned matrix $\mathcal{P}_s^{-\frac{1}{2}} \mathcal{M} \mathcal{P}_s^{-\frac{1}{2}}$ lie in the range $\mathcal{I}^- \cup \mathcal{I}^+ \subset \mathbb{R}$, with*

$$\mathcal{I}^+ = \left[\frac{\min\{\Gamma_{\tilde{A}}, \gamma_{\tilde{S}}\}(2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2})}{2}, \ \mathcal{U} \right] \subset \mathbb{R}^+,$$

$$\mathcal{I}^- = \left[-\mathcal{V}, \ -\frac{\min\{\Gamma_{\tilde{S}}, \gamma_{\tilde{A}}\}(2 + \ell_1 - \sqrt{4\ell_1 + \ell_1^2})}{2} \right] \subset \mathbb{R}^-.$$

Remark 5.1. We can easily see that $\ell_1 \approx \ell_2$ if the preconditioner $\hat{A}$ approximates A well, by noting that $\tilde{H}^T \tilde{H} = \hat{S}^{-\frac{1}{2}}(B\hat{A}^{-1}B)\hat{S}^{-\frac{1}{2}}$ and $\tilde{H}^T \tilde{A}^{-2} \tilde{H} = \hat{S}^{-\frac{1}{2}}(BA^{-1}\hat{A}A^{-1}B)\hat{S}^{-\frac{1}{2}}$.

Table 1
Numerical results of the MINRES method.

DOF	No preconditioning			Preconditioner $\mathcal{P}_s$		
	CPU	ERR	Iter	CPU	ERR	Iter
1 360	0.2778	9.9813e−06	2 859	0.1460	8.4774e−06	109
5 600	4.1394	9.9997e−06	15 364	0.7044	8.5037e−06	161
22 720	74.3719	9.9998e−06	58 235	5.5655	8.5590e−06	244

Table 2
Numerical results of the MINRES method.

DOF	Preconditioner $\mathcal{P}_0$			Preconditioner $\mathcal{P}_1$			Preconditioner $\mathcal{P}_2$		
	CPU	ERR	Iter	CPU	ERR	Iter	CPU	ERR	Iter
1 360	0.0215	9.9676e−06	135	0.0731	9.4188e−06	136	0.0752	9.2952e−06	137
5 600	0.1103	9.9713e−06	198	0.3997	9.6107e−06	198	0.4035	9.2047e−06	201
22 720	1.3845	9.9699e−06	547	4.7839	9.9990e−06	547	4.9255	9.9400e−06	563

6. Numerical experiments

In this section, we present some numerical results for the preconditioners we introduced in Sections 3 and 4. All experiments were run on a PC with Intel(R) Core(TM) i7-4770 CPU @3.40 GHz 16 GB, and all programmings are implemented in MATLAB R2014a. In our experiments, we take the following data for the scattering system (1.1) as it is done in [1,2]:

$$\boldsymbol{g} = \nabla \times [H_1^{(1)}(r)e^{i\theta}]|_{\Gamma_0}, \quad \Omega_0 = (-1, 1)^2, \quad \mu = 1, \quad \varepsilon = 1, \quad k = 1,$$

where $H_1^{(1)}(r)$ is the Hankel function of first kind. Then the analytic solution to the system (1.1) is given by $\boldsymbol{u} = \nabla \times [H_1^{(1)}(r)e^{i\theta}]$. As we did in Section 2, we may derive the PML edge element system (1.2) of the scattering system (1.1).

In our all implementations, we take the initial guesses to be the zero vectors and terminate the concerned algorithms when the relative residual ERR $\leq 10^{-5}$ or the number of iteration is greater than 4000, where ERR is given by

$$\text{ERR} := \frac{\|b - \mathcal{M}z^k\|}{\|b\|},$$

where $z^k = ((x^k)^T, (y^k)^T)^T$ is the kth iterative solution of the system (1.2). We shall compare the performance of several iterative methods by reporting the number of iterations, the total CPU time, the degree of freedom, and the relative residual error, which are respectively denoted by "Iter", "CPU", "DOF" and "ERR". As it is motivated by [1], we take two symmetric positive definite approximations $\hat{A}$ and $\hat{S}$ of matrix A and Schur complement $S = A + BA^{-1}B$ to be the stiffness matrices induced by the following bilinear forms

$$\int_{\Omega_L} |\alpha|(\nabla \times \boldsymbol{u})(\nabla \times \boldsymbol{v})\mathrm{d}\boldsymbol{x} + k^2 \int_{\Omega_L} \boldsymbol{u}^T D\boldsymbol{v}\mathrm{d}\boldsymbol{x}, \ \forall \, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_{h,0}(\Omega_L),$$

$$\int_{\Omega_L} \hat{\alpha}(\nabla \times \boldsymbol{u})(\nabla \times \boldsymbol{\psi})\mathrm{d}\boldsymbol{x} + k^2 \int_{\Omega_L} \boldsymbol{u}^T \hat{D}\boldsymbol{\psi}\mathrm{d}\boldsymbol{x}, \ \forall \, \boldsymbol{u}, \boldsymbol{\psi} \in \mathbb{V}_{h,0}(\Omega_L),$$

respectively, where Ω_L is taken to be $\Omega_L = (-4, 4)^2 \setminus [-1, 1]^2$.

Let $\mathcal{P}_s$ be the preconditioner (1.3), and $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are the preconditioners we have proposed in (3.1), with $\eta = 0, 0.01, 0.1$ and $L = \text{ichol}(\hat{A})$:

$$\mathcal{P}_0 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \quad \mathcal{P}_1 = \begin{pmatrix} L & 0 \\ 0.01L & L \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} L & 0 \\ 0.1L & L \end{pmatrix}.$$

Then we shall solve the edge element PML system (1.2), using respectively the MINRES method (with no preconditioning), the preconditioned MINRES method with 4 preconditioners $\mathcal{P}_s$, $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$, the GMRES method (with no preconditioning), and the preconditioned GMRES method with the block triangular preconditioner $\tilde{\mathcal{P}}$ in (4.1).

The numerical results are listed in Tables 1–3 and Figs. 1–2 for all the aforementioned methods. In Table 1 and Fig. 1, we can see that the MINRES method with no preconditioning is quite impractical and expensive in terms of the CPU times and numbers of iterations, but the preconditioner $\mathcal{P}_s$ can essentially improve its performance. Similarly, from Table 3 and Fig. 2 we can see that the GMRES with no preconditioning does not work well for solving the system (1.2), while the block triangular preconditioner $\tilde{\mathcal{P}}$ can greatly improve its performance.

In comparison with preconditioner $\mathcal{P}_s$, we can observe from Tables 1 and 2 that preconditioners $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ require much more iterations but less CPU times.

Next, we will check the effectiveness and reliability of the estimates we have obtained in this work. To do so, we choose the scattering system (1.1) with DOF $= 1360$ to examine the results in Theorems 3.1 and 4.3 respectively for the preconditioners

Table 3
Numerical results of the GMRES method.

DOF	No preconditioning			Preconditioner $\tilde{\mathcal{P}}$		
	CPU	ERR	Iter	CPU	ERR	Iter
1 360	5.5962	9.9124e−06	866	0.7327	8.5160e−06	110
5 600	445.5941	9.8611e−06	3488	4.9807	9.6847e−06	146
22 720	–	–	–	44.3358	9.1143e−06	237
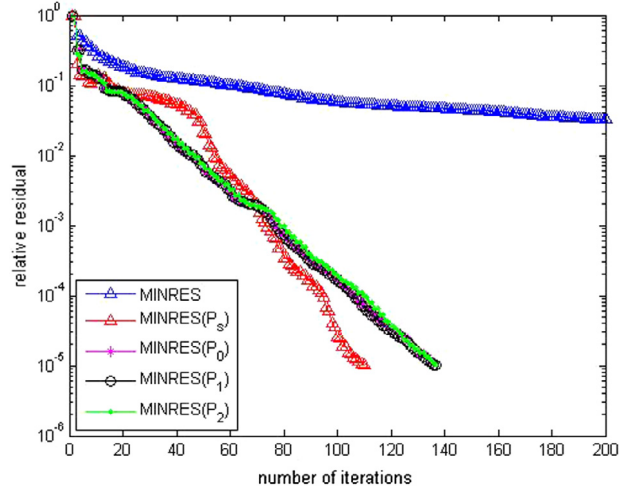
Fig. 1. Iterations vs. errors for MINRES with DOF = 1360.
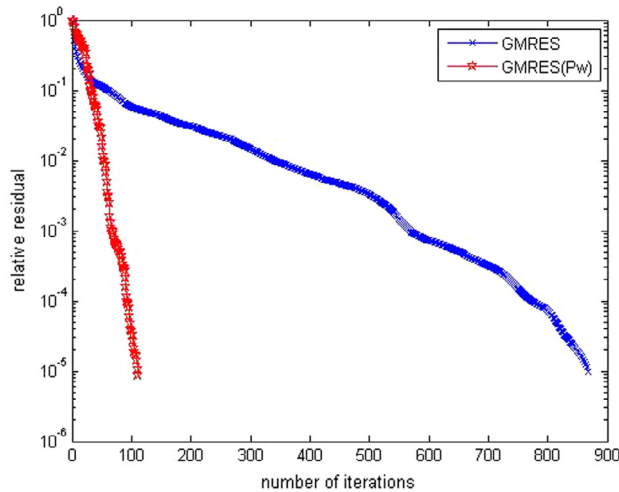
Fig. 2. Iterations vs. errors for GMRES with DOF = 1360.

$\mathcal{P}$ and $\tilde{\mathcal{P}}$ in (3.1) and (4.1). By using Matlab, we have computed the exact values of $\|\tilde{B}\|_2$, $\mathrm{sp}(\tilde{A})$, $\mathrm{sp}(\tilde{S}^{-1})$ and ν defined in Lemma 3.4 as follows:

$$\|\tilde{B}\|_2 = 3.9837, \qquad \nu = 7.9024,$$

and

$$\mathrm{sp}(\tilde{S}^{-1}) \subset [-1.6176, 2.6112], \qquad \mathrm{sp}(\tilde{A}) \subset [-1.0000, -0.0030] \cup [0.0233, 0.9848].$$

The estimates derived by Theorem 3.1 can be seen in Tables 4–6.

Table 4
The bounds on the spectra of $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ with $\eta = 0.1$.

Exact value	$[-4.5936, -0.2177] \cup [0.2514, 3.9785]$
Estimated value	$[-8.1590, -0.1227] \cup [0.1227, 8.1590]$

Table 5
The bounds on the spectra of $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ with $\eta = 0.01$.

Exact value	$[-4.1236, -0.2425] \cup [0.2471, 4.0467]$
Estimated value	$[-5.2310, -0.1914] \cup [0.1914, 5.2310]$

Table 6
The bounds on the spectra of $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ with $\eta = 0$.

Exact value	$[-4.0835, -0.2449] \cup [0.2449, 4.0835]$
Estimated value	$[-4.9761, -0.2012] \cup [0.2012, 4.9761]$

As can be seen in Tables 4–6, Theorem 3.1 provides quite reliable and accurate spectral bounds for the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$. And we can also see from these tables that the preconditioner $\mathcal{P}^{-1}$ is quite effective as the preconditioned system $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}^{-T}$ is not very ill-conditioned.

For the spectral estimate of the preconditioned system $\tilde{\mathcal{P}}^{-1}\mathcal{M}$, we see $\Lambda_{\tilde{A}} = 0.9848 < 1$, hence we know from Theorem 4.3 that all the eigenvalues of $\tilde{\mathcal{P}}^{-1}\mathcal{M}$ are real, and their predicted estimates lie in the range $[-18.6529, 1.0000]$. Then we have also computed the exact bounds of the eigenvalues of $\tilde{\mathcal{P}}^{-1}\mathcal{M}$, given by $[-16.8578, 1.0000]$. This shows our estimates in Theorem 4.3 are rather effective and accurate. To further check the effectiveness of our theoretical estimates in Theorem 4.3, we replace the approximation $\hat{A}$ of A used in the preconditioner $\tilde{P}$ in (4.1) by a more crude approximation, namely $2\hat{A}$. For this case, we can compute $\Lambda_{\tilde{A}} = 0.4924 < 1$, $\lambda_{\tilde{A}} = 0.5000 < 1$ and $\|\tilde{B}\|_2^2 = 3.9675$, which result in the predicted spectral estimate by Theorem 4.3 as $\mathrm{sp}(\mathcal{P}^{-1}\mathcal{M}) \subset [-5.6219, 0.9485]$. This is very close to the exact spectral bound $[-4.8296, 0.8669]$, so indicates again the effectiveness of our theoretical spectral estimates.

Acknowledgments

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