MATH 2060 Mathematical Analysis II
some examples related with uniform convergence

Definition 0.1. *f_n is said to be convergent to f on $[a,b]$ if for all $x \in [a,b]$, $\epsilon > 0$, there exists $N = N(x,\epsilon)$ such that for all $n > N$, $|f_n(x) - f(x)| < \epsilon$.*

Definition 0.2. *We say that f_n converge uniformly to f on $[a,b]$ if for all $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that for all $n > N$, $|f_n(x) - f(x)| < \epsilon$.*

Proposition 0.1. *If $f_n \in R[a,b]$, f_n converge uniformly to f, then $f \in R[a,b]$.*

Proposition 0.2. *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of real valued function which converge to f uniformly on $[a,b]$. Furthermore, suppose there exists $M > 0$ such that $|f_n| \le M$ for all $n \in \mathbb{N}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $g \circ f_n$ converge uniformly to $g \circ f$ on $[a,b]$.*

Proof. By uniform continuity theorem, g is uniform continuous on $[-M,M]$. Let $\epsilon > 0$, there exists $\delta > 0$ such that for all $x,y \in [-M,M]$ and $|x-y| < \delta$, we have $|g(x) - g(y)| < \epsilon$.
On the other hand, there exists $N = N_\delta$ such that for all $n > N$, $|f_n(x) - f(x)| < \delta \ \ \forall x \in [a,b]$.
Therefore, for any $x \in [a,b]$, $n > N$, we can conclude that

$$|g \circ f_n(x) - g \circ f(x)| < \epsilon.$$

As N depends on δ and the continuity of g only, the convergence is uniform. □

Example 0.3. *(Mid-term Question) If $f \ge 0$ is a Riemann integrable function on $[0,1]$, then $\sqrt{f}$ is also Riemann integrable.*

Proof. Suppose f is bounded below by a positive constant $\delta > 0$, then the situation is easy. For the sake of completeness, we include the proof here.
Let $\epsilon > 0$, there exists a partition $\mathcal{P}$ on $[0,1]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < 2\epsilon\sqrt{\delta}.$$

Then for such partition $\mathcal{P}$,

$$U(\sqrt{f},\mathcal{P}) - L(\sqrt{f},\mathcal{P}) \le \frac{1}{2\sqrt{\delta}}\Big[U(f,\mathcal{P}) - L(f,\mathcal{P})\Big] < \epsilon.$$

The first inequality is due to the fact that

$$|\sqrt{f(x)} - \sqrt{f(y)}| = \left|\frac{f(x) - f(y)}{\sqrt{f(x)} + \sqrt{f(y)}}\right| \le \frac{|f(x) - f(y)|}{2\sqrt{\delta}}.$$

However if f is only bounded below by 0, the estimation is a bit more technical as shown in the tutorial class. Instead of doing this, we may consider $f_n = f + \frac{1}{n}$. And let $g(x) = \sqrt{x}$. By above arguement, $\sqrt{f_n} = g \circ f_n \in R[0,1]$. And f_n is bounded uniformly because of the fact that $f \in R[0,1]$. Clearly, f_n converge uniformly to f. Hence $g \circ f_n$ converge uniformly to $g \circ f = \sqrt{f}$ as well. By the first proposition, this implies that $g \circ f \in R[0,1]$ which is our desired result.

□