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Crystals and Crystalline Cohomology

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Exercise sheet 7[1]

Definition 1. *Suppose that we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & Y \arrow[d, "f"] \\ S_0 \arrow[r] & S \end{tikzcd}$$

in which i is a closed immersion and f is smooth. Let D be the p-adic completion of the PD-envelope of $i : X \hookrightarrow Y$ with respect to $(S, \mathcal{I}, \gamma)$. A quasi-nilpotent connection associated with this diagram is a pair (M, ∇) satisfying the following conditions:

(1) M is a quasi-coherent $\mathcal{O}_D$-module and it is p-adically complete;

(2) $\nabla : M \to M \widehat{\otimes}_{\mathcal{O}_D} \Omega_D$ is a connection of D/S in the sense that it is $\mathcal{O}_S$-linear and satisfies the Leibniz rule. Here Ω_D denotes the PD-differential $\Omega^1_{D/S,\bar{\gamma}}$, see `https://stacks.math.columbia.edu/tag/07HR` *for the definition of $\Omega^1_{D/S,\bar{\gamma}}$;*

(3) ∇ is integrable in the obvious sense;

(4) ∇ is topologically quasi-nilpotent: For any section s on an affine open $U \subseteq D$ which admits a system of local coordinates

$$\{dx_1, \cdots, dx_n\}$$

coming from Y, there exists $k \in \mathbb{N}$ such that $(\frac{\partial}{\partial x_i})^k(s) \in pM$.

We denote the category of quasi-nilpotent connections with respect to the diagram above by $\mathrm{QNC}(Y/S)$.

Exercise 1. Show that the category of quasi-nilpotent connections in quasi-coherent modules $\mathrm{QNC}(Y/S)$ is equivalent to the category of quasi-coherent crystals on $\mathrm{Cris}(X/S)$.

[1] If you have any questions concerning these exercises you can contact me via l.zhang@fu-berlin.de.