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Crystals and Crystalline Cohomology

Dr. Lei Zhang

Exercise sheet 6[1]

Exercise 1. In the class we started with an algebraic C over A/I, where (A, I, δ) is a PD-triple. Then we take a surjection of A-algebras $P = A[x_i] \twoheadrightarrow C$, and define J to be its kernel. We denote D the PD-envelope of $(A, I, \gamma) \to (P, J)$. Similarly we take

$$\underbrace{P \otimes_A \cdots \otimes_A P}_{(n+1)-\text{times}} \twoheadrightarrow C$$

and denote $J(n)$ its kernel. We take $D(n)$ to be the PD-envelope of $(A, I, \gamma) \to (P \otimes_A \cdots \otimes_A P, J(n))$. In this way the projections $p_i \colon P \to P \otimes_A \cdots \otimes_A P$ induces projections $p_i \colon D \to D(n)$ and the diagonal $P \otimes_A \cdots \otimes_A P \to P$ induces the diagonal map $D(n) \to D$. In the class we started with a crystal $\mathcal{F}$ on $\mathrm{Cris}(C/A)$, and we take its values on $(X, \mathrm{Spec}\,(D_e), \bar{\gamma}_e)$ or on $(X, \mathrm{Spec}\,(D(n)_e), \bar{\gamma}_e)$. We denote the values M_e and $M(n)_e$ respectively. Since $\mathcal{F}$ is a crystal, we have the isomorphisms

$$M \otimes_{D,p_0} D(1) \longrightarrow M(1) \longleftarrow M \otimes_{D,p_1} D(1)$$

Show that if we pullback the isomorphism along the diagonal $D(1) \to D$, then we get the identity $\mathrm{id} \colon M \to M$.

Remark 1. Recall that for $m \in M$ we can always write

$$c(m \otimes 1) = \sum_K \theta_K(m) \otimes \prod \xi^{[k_i]}$$

where those K are multi-indices $(k_i)_{i \in I}$ and $\sum_{i \in I} k_i < \infty$. The exercise indeed tells us that that $\theta_0 \colon M \to M$ is the identity.

[1]If you have any questions concerning these exercises you can contact me via l.zhang@fu-berlin.de.