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Crystals and Crystalline Cohomology

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Exercise sheet 2[1]

Exercise 1. Let B be an A-algebra, and let M be a B-module. We define a connection on M to be an A-linear map-linear map:

$$\nabla \colon M \longrightarrow M \otimes_B \Omega_{B/A}$$

which satisfies the Leibniz rule: $\nabla(bm) = b\nabla(m) + m \otimes db$. Show that start with a connection we get canonically a sequence of A-linear maps:

$$M \xrightarrow{\nabla} M \otimes_B \Omega_{B/A} \xrightarrow{\nabla^1} M \otimes \Omega^2_{B/A} \xrightarrow{\nabla^2} \cdots$$

We say the connection on M is integrable if we have $\nabla^1 \circ \nabla = 0$. In this case the sequence of maps we obtained is actually a complex, and it is called the de Rham complex associated with the connection.

Exercise 2. Show that if $B \to B'$ is a map of A algebras, and if M is a B-module equipped with a connection ∇, then there is a canonical connection ∇' on $M \otimes_B B'$. Moreover, if (M, ∇) is integrable, then so is $(M \otimes_B B', \nabla')$, and in this there is a map of chain complexes (of abelian groups)

$$M \otimes_B \Omega^\bullet_{B/A} \longrightarrow M \otimes_B \Omega^\bullet_{B'/A}$$

Exercise 3. Show that if (A, I, γ) is a PD-triple, and if B is a flat A-algebra, then there exists a PD-structure δ on IB such that the nature map $(A, I) \to (B, IB)$ is indeed a PD-morphism $(A, I, \gamma) \to (B, IB, \delta)$. (Hint: We know that the PD-envelop is stable under flat base change. Thus the PD-envelop of (B, IB) is the base change of the PD-envelop of (A, I) to B.)

Exercise 4. Let (B, J, δ) be a PD-triple, where B is an A-algebra. Let p be a prime number. Assume that A is a $\mathbb{Z}_{(p)}$-algebra and p is nilpotent on B/J. Then we have canonical isomorphisms

$$\varprojlim_e \Omega_{B_e/A,\delta_e} \sim \varprojlim_e \Omega_{B/A,\delta}/p^e\Omega_{B/A,\delta} \sim \varprojlim_e \Omega_{\hat{B}/A,\hat{\delta}}/p^e\Omega_{\hat{B}/A,\hat{\delta}}$$

where δ_e is the induced PD-structure on $(B_e \coloneqq B/p^eB, JB_e)$ and $\hat{B}$ is the p-adic completion of B.

[1] If you have any questions concerning these exercises you can contact me via l.zhang@fu-berlin.de.