Crystals and Crystalline Cohomology

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Exercise sheet 1^1

Exercise 1. Let (A, I, γ) be a PD-triple. Let M be an A-module. Let $B = A \oplus M$ as an A-algebra where M is an ideal of square zero. Set $J := I \oplus M$, and set

$$\delta_n(x+z) \coloneqq \gamma_n(x) + \gamma_{n-1}(x)z$$

for all $x \in I$ and $z \in M$. Show that δ is a PD-structure and $A \to B$ is a homomorphism of PD-triples.

Exercise 2. Let $P \to C$ be a surjection of A-algebras with kernel J. We write $(D, \overline{J}, \overline{\gamma})$ for the PD-envelop of (P, J) with respect to (A, I, γ) . Let $(\hat{D}, \overline{J}, \overline{\hat{\gamma}})$ be the completion of $(D, \overline{J}, \overline{\gamma})$. For every $e \geq 1$, set $(P_e, J_e) := (P/p^e P, J/(J \cap p^e P))$ and $(D_e, \overline{J}_e, \overline{\gamma}_e)$ the PD-envelop of this pair. Show that for large e we have

- (1) $p^e D \subseteq \overline{J}$ and $p^e \hat{D} \subseteq \hat{J}$ are preserved by the PD-structures.
- (2) $\hat{D}/p^e \hat{D} \cong D/p^e D = D_e$ as PD-rings.
- (3) $(D_e, \overline{J}_e, \overline{\gamma}_e) \in \operatorname{Cris}(C/A).$
- (4) $(\hat{D}, \hat{\bar{J}}, \bar{\bar{\gamma}}) = \underline{\lim}(D_e, \bar{J}_e, \bar{\gamma}_e).$
- (5) $(\hat{D}, \hat{\bar{J}}, \hat{\bar{\gamma}}) \in \widehat{\operatorname{Cris}}(C/A).$

Exercise 3. Let P be a polynomial algebra over A, and let $P \twoheadrightarrow C$ be a surjection of A-algebras with kernel J. Show that for every object (B, J, δ) of CRIS(C/A) there exists an e and a morphism

$$(D_e, \bar{J}_e, \bar{\gamma}_e) \to (B, J, \delta)$$

in $\operatorname{CRIS}(C/A)$.

Exercise 4. Let P be a polynomial algebra over A, and let $P \twoheadrightarrow C$ be a surjection of A-algebras with kernel J. Let $(D, \overline{J}, \overline{\gamma})$ be the p-adic completion of $D_{P,\gamma}(J)$. Show that for every object (B, J, δ) of $\widehat{\operatorname{Cris}}(C/A)$ there exists a morphism

$$(D, J, \bar{\gamma}) \to (B, J, \delta)$$

in Cris(C/A).

 $^{^1\}mathrm{If}$ you have any questions concerning these exercises you can contact me via <code>l.zhang@fu-berlin.de</code>.