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A property of Pisot numbers ☆

De-Jun Feng* and Zhi-Ying Wen

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China

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Abstract

Let q be a Pisot number and m a positive integer. Consider the increasing sequence

$$0 = y_0 < y_1 < \cdots < y_k < \cdots$$

of those real numbers y which have at least one representation of the form

$$y = \sum_{i=0}^{n} \varepsilon_i q^i$$

with some integer $n \geqslant 0$ and coefficients $\varepsilon_i \in \{0, 1, \ldots, m\}$. When $m \geqslant q - 1$, we will determine the structure of the difference sequence $\{y_{k+1} - y_k\}_{k \geqslant 0}$, that is, it is the image of a sequence generated by a substitution over a finite alphabet of symbols. Then, we also give an algorithm to determine the exact value of $\inf_k (y_{k+1} - y_k)$.

MSC: primary 11R06; secondary 11B85

Keywords: Pisot numbers; Substitutions; Algorithms

1. Introduction

Fix a real number $q > 1$ and a positive integer m. Consider the increasing sequence

$$0 = y_0 < y_1 < \cdots < y_k < \cdots \tag{1.1}$$

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*Corresponding author.

E-mail addresses: dfeng@math.tsinghua.edu.cn (D.-J. Feng), wenzy@tsinghua.edu.cn (Z.-Y. Wen).

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of those real numbers y which have at least one representation of the form

$$y=\sum_{i=0}^{n}\varepsilon_i q^i$$

with some integer $n\geqslant 0$ and coefficients $\varepsilon_i\in\{0,1,\dots,m\}$. A question initiated by Erdös et al. [EJK1] is to analyze the properties of the difference sequence $\{y_{k+1}-y_k\}_{k\geqslant 0}$.

Erdös and Komornik [EK] distinguished the cases $m\geqslant q-1$ and $m<q-1$:

Lemma 1.1. *Let $q>1$ and m be a positive integer. Then we have*:

(i) *If $m\geqslant q-1$, then $y_{k+1}-y_k\leqslant 1$ for each k*;
(ii) *If $m<q-1$, then there exists a subsequence $\{k_n\}$, such that $y_{k_n+1}-y_{k_n}$ tends to infinity.*

Remark 1.2. Statement (ii) above is only implied in the proof of Lemma 2.1 in [EK].

To characterize this difference sequence, Bugeaud [B] proved that $\liminf(y_{k+1}-y_k)\neq 0$ for every integer $m\geqslant 1$ if and only if q is a Pisot number. Recall that $q>1$ is called a Pisot number if q is an algebraic number and all its conjugates have moduli less than 1 (see [BDGPS,Sa] for detailed properties of Pisot numbers). In his proof, Bugeaud used some results of automata in [BF,F]. One may see [EJK2] or [EK] for a different proof.

As we will prove in Section 2 (see Lemma 2.2), for a Pisot number q with $m\geqslant q-1$ the difference sequence $\{y_{k+1}-y_k\}_{k\geqslant 0}$ can take only finitely many distinct values. It is then natural to ask how to describe the structure of this sequence. The first aim of this paper is to prove by a concrete construction that this difference sequence can be generated by a substitution over a finite alphabet (see Theorem 2.1). Sequences generated by substitutions have many interesting properties. In particular, it is related closely to number theory (see [Al] or [Sh] for a survey).

After Erdös et al. [EJK1], several authors [EJJ,KLP] determined the exact value of $\inf_k(y_{k+1}-y_k)$ for some special Pisot numbers q and integers m ($\geqslant q-1$) (see Section 3 for details). The method, which they used, depends on the choice of q. They asked whether one can determine $\inf_k(y_{k+1}-y_k)$ for the general case. The second aim of this paper is to give an algorithm for the general case. At the end of this paper, we will also give some examples and computation results.

2. Structure of the sequence of $\{y_{k+1} - y_k\}_{k \geqslant 1}$

In this section, we will prove that the sequence $\{y_{k+1} - y_k\}_{k \geqslant 1}$ can be generated by a substitution over a finite alphabet. For this purpose, we recall first some definitions.

Let $\mathscr{A}$ be a finite nonempty set. The set $\mathscr{A}$ is also called an *alphabet* and its elements *letters*. The free monoid generated by $\mathscr{A}$ is denoted by $\mathscr{A}^*$; it contains all finite *words*, i.e., finite strings of symbols from $\mathscr{A}$, including the empty word. The *length* of a word w, denoted by $|w|$, is defined as the number of letters of w. The operation which makes $\mathscr{A}^*$ a monoid is the *concatenation* of words.

A substitution on $\mathscr{A}^*$ is a map $\sigma : \mathscr{A}^* \to \mathscr{A}^*$, such that, for any two words u and v, one has $\sigma(uv) = \sigma(u)\sigma(v)$. The substitution σ is determined by the image of the elements of $\mathscr{A}$. We say σ is of *constant length* if the length of each word $\sigma(a)$ is equal to a constant for $a \in \mathscr{A}$.

Denote by $\mathscr{A}^{\mathbb{N}}$ the collection of all infinite words over $\mathscr{A}$. A substitution σ on $\mathscr{A}^*$ can be uniquely extended to a map (denoted also by σ) on $\mathscr{A}^{\mathbb{N}}$ in a natural way. A sequence $y \in \mathscr{A}^{\mathbb{N}}$ is called a *fixed point* of the substitution σ if $\sigma(y) = y$. If for some letter $a \in \mathscr{A}$, the word $\sigma(a)$ begins at a and has length at least 2, then the sequence of words $\sigma^n(a)$ converges to a fixed point $\sigma^\infty(a) \in \mathscr{A}^{\mathbb{N}}$.

Let $\mathscr{B}$ be an alphabet. A sequence $x = x_0 x_1 \cdots x_n \cdots \in \mathscr{B}^{\mathbb{N}}$ is called *substitutive* if there exists a fixed point $\omega = \omega_0 \omega_1 \cdots \omega_n \cdots \in \mathscr{A}^{\mathbb{N}}$ of a substitution σ over an alphabet $\mathscr{A}$ and a map $h : \mathscr{A} \to \mathscr{B}$ such that $x_i = h(\omega_i)$, $i \geqslant 0$, i.e., x is the image of ω under h. In this case, we also say that the sequence x is generated by the substitution σ. The reader is referred to Allouche [Al] and Queffélec [Q] for further properties of substitutions.

Now we can formulate our result as follows.

Theorem 2.1. *Fix a Pisot number $q > 1$ and a positive integer m. Let $\{y_k\}_{k \geqslant 0}$ be the sequence defined as in* (1.1). *If $m \geqslant q - 1$, the difference sequence $\{y_{k+1} - y_k\}_{k \geqslant 0}$ is the image of a substitution sequence over a finite alphabet of symbols. If $m < q - 1$,* $\limsup(y_{k+1} - y_k) = \infty$.

The above theorem generalizes a result of Bugeaud [B1], who considered the case $m = 1$ and obtained the corresponding substitution property for a special sequence of Pisot numbers in $(1, 2)$ (see Remark 2.6 for details). The proof of Theorem 2.1 is based on the following lemmas.

Lemma 2.2. *Let q be a Pisot number and m a positive integer so that $m \geqslant q - 1$. Then the sequence $\{y_{k+1} - y_k\}_{k \geqslant 0}$ only takes a finite number of* *values, and this finite number is not greater than*

$$\frac{(2m)^{d-1}}{\prod_{i=1}^{d-1}(1-|\alpha_i|)}, \tag{2.1}$$

where d is the degree of q and $\alpha_1, \ldots, \alpha_{d-1}$ are the algebraic conjugates of q.

Remark 2.3. The finiteness of the number of distinct values of $(y_{k+1} - y_k)$ was proved by Bugeaud [B] for Pisot numbers $1 < q < 2$.

Proof of Lemma 2.2. Since q is a Pisot number, a classic result of algebraic number theory by Garsia (see [G, Lemma 1.51]) states that, if $A(x)$ is a polynomial with integer coefficients and height M for which $A(q) \neq 0$, then

$$|A(q)| \geqslant \frac{\prod_{i=1}^{d-1}(1-|\alpha_i|)}{M^{d-1}}. \tag{2.2}$$

Now assume that our lemma is not true, that is, the sequence $\{y_{k+1} - y_k\}$ takes at least N distinct values, where N is strictly greater than $(2m)^{d-1}/\prod_{i=1}^{d-1}(1-|\alpha_i|)$.

Since $y_{k+1} - y_k \leqslant 1$ for any k by Lemma 1.1(i), we see that either

$$0 < y_{k+1} - y_k < \frac{1}{N}$$

for some k, or, by Pigeon-hole Principle,

$$0 < (y_{k'+1} - y_{k'}) - (y_{k''+1} - y_{k''}) \leqslant \frac{1}{N}$$

for some k' and k''. However due to (2.2) this cannot hold since $y_{k+1} - y_k$ or $(y_{k'+1} - y_{k'}) - (y_{k''+1} - y_{k''})$ is equal to $B(q)$ for some polynomial $B(x)$ with integer coefficients and height not exceeding $2m$. □

Proof of Theorem 2.1. Let $q > 1$ be a Pisot number and m a positive integer. By Lemma 1.1, $\limsup(y_{k+1} - y_k) = \infty$ whenever $m < q - 1$. In the following we assume that q is not an integer and $m > q - 1$ (the case where q is an integer is trivial, since the sequence $(y_{k+1} - y_k)$ takes the constant value 1 when $m > q - 1$). For each positive integer n, let

$$0 = z_{n,0} < z_{n,1} < \cdots < z_{n,s_n}$$

be all the distinct elements of the set

$$\Xi_n = \left\{ \sum_{i=0}^{n-1} \varepsilon_i q^i : \varepsilon_i \in \{0, 1, \ldots, m\} \right\}$$

and denote $z_{n,s_n+1} = z_{n,s_n} + \frac{m}{q-1} = \frac{mq^n}{q-1}$.

For each $i \in \{0, 1, \dots, s_n\}$, the segment $[z_{n,i}, z_{n,i+1}]$ is termed as an nth *net interval*, for which we construct a set $\gamma_{n,i}$ by

$$\gamma_{n,i} = \left\{ z_{n,j} - z_{n,i} : 0 \leqslant j \leqslant i+1, z_{n,j} - z_{n,i} \geqslant -\frac{m}{q-1} \right\}.$$

We term $\gamma_{n,i}$ as the nth *color* of the segment $[z_{n,i}, z_{n,i+1}]$. In particular, we denote

$$\|\gamma_{n,i}\| = z_{n,i+1} - z_{n,i}, \tag{2.3}$$

that is, $\|\gamma_{n,i}\|$ represents the length of $[z_{n,i}, z_{n,i+1}]$.

Let $\mathscr{A}$ be the collection of all the sets $\gamma_{n,i}$, that is,

$$\mathscr{A} = \{\gamma_{n,i} : n \geqslant 1,\ \ i \in \{0, 1, \dots, s_n\}\}.$$

Using an argument similar to the proof of Lemma 2.2, we can show that $\mathscr{A}$ has only finitely many elements. To see this, suppose that q is of degree d and $\alpha_1, \dots, \alpha_{d-1}$ are its algebraic conjugates. If $z_{n,j} - z_{n,j'} \neq 0$ for some n and $j, j' \in \{0, 1, \dots, s_n\}$, by (2.2),

$$|z_{n,j} - z_{n,j'}| \geqslant \frac{\prod_{i=1}^{d-1} (1 - |\alpha_i|)}{m^{d-1}},$$

thus each set $\gamma_{n,i}$ consists of at most $N_1 = \frac{m^d}{(q-1)\prod_{i=1}^{d-1}(1-|\alpha_i|)} + 2$ elements. On the other hand, the cardinality of the following set:

$$\left\{ |z_{n,j} - z_{n,j'}| : |z_{n,j} - z_{n,j'}| \leqslant \frac{m}{q-1},\ \ n \geqslant 0,\ \ j, j' \in \{0, 1, \dots, s_n\} \right\}$$

does not exceed $N_2 = \frac{m}{q-1} \frac{(2m)^{d-1}}{\prod_{i=1}^{d-1}(1-|\alpha_i|)}$. Therefore, the cardinality of $\mathscr{A}$ does not exceed $N_1 N_2^{N_1}$. Now, take an nth net interval $J = [a, b]$ arbitrarily, and denote $qJ = [qa, qb]$. It is clear that the endpoints of qJ are contained in the set $\Xi_{n+1} \cup \{\frac{mq^{n+1}}{q-1}\}$, and thus qJ is the union of some $(n+1)$th net intervals. That is, $\exists \{j, j+1, \dots, j+l-1\} \subset \{0, 1, \dots, s_{n+1}\}$, such that

$$qJ = \bigcup_{k=1}^{l} J_k, \tag{2.4}$$

where $J_k = [z_{n+1,j+k-1}, z_{n+1,j+k}]$. Let ξ_k be the $(n+1)$th color of J_k, $k = 1, \dots, l$. As a word in $\mathscr{A}^* = \bigcup_{n=1}^{\infty} \mathscr{A}^n$, $\xi_1 \cdots \xi_l$ is determined completely by the nth color, say ξ, of J. To see this, let $\xi = \{t_1, t_2, \dots, t_r\}$, where $t_1, \dots, t_r$ are in the increasing order. Then by the definition of the nth color,

$$\left[a - \frac{m}{q-1}, b\right) \cap \Xi_n = \{a + t_1, \dots, a + t_{r-1}\}, \quad b - a = t_r.$$

Since $\Xi_{n+1} = \bigcup_{i=0}^{m} (q\Xi_n + i)$, a direct check shows that

$$\left[qa - \frac{m}{q-1}, qb\right) \cap \Xi_{n+1} = \left\{qa + qt_j + i : qt_j + i \in \left[-\frac{m}{q-1}, qt_r\right), 1 \leqslant j \leqslant r-1, 1 \leqslant i \leqslant m\right\} \quad (2.5)$$

and

$$[qa, qb) \cap \Xi_{n+1} = \{qa + qt_j + i : qt_j + i \in [0, qt_r), 1 \leqslant j \leqslant r-1, 1 \leqslant i \leqslant m\}. \quad (2.6)$$

By (2.6), the number of $(n+1)$th net intervals contained in qJ, denoted by l in (2.4), is equal to the cardinality of the set

$$\left\{qt_j + i : qt_j + i \in \left[-\frac{m}{q-1}, qt_r\right),\ 1 \leqslant j \leqslant r-1,\ 1 \leqslant i \leqslant m\right\}.$$

Thus l only depends on ξ. Now for any integer $1 \leqslant k \leqslant l$, the $(n+1)$th color ξ_k of $J_k = [z_{n+1,j+k-1}, z_{n+1,j+k}]$ can be written as

$$\left\{w - z_{n+1,j+k-1} : w \in \left[qa - \frac{m}{q-1}, qb\right) \cap \Xi_{n+1}, -\frac{m}{q-1} \leqslant w - z_{n+1,j+k-1} \leqslant 0\right\} \cup \{z_{n+1,j+k} - z_{n+1,j+k-1}\}.$$

By (2.5) and (2.6), the differences $w - z_{n+1,j+k-1}$ are independent of a, thus ξ_k only depends upon ξ. By the above process, we get a map $\sigma : \mathscr{A} \to \mathscr{A}^*$ defined by $\sigma(\xi) = \xi_1 \cdots \xi_l$, where l varies depending on ξ. In other words, we get a substitution σ on $\mathscr{A}$.

One can check that $[0,1]$ is a 1st interval with the 1st color $\theta = \{0,1\}$, and $\sigma(\theta) = \theta_1\theta_2 \cdots \theta_{[q]+1}$ (here and in the following formula we use $[x]$ to denote the integral part of the number x), where

$$\begin{cases} \theta_i = \{a_i, a_i+1, \ldots, 0, 1\}, & a_i = \max\{-i+1, -[\frac{m}{q-1}]\} \\ & \text{for } 1 \leqslant i \leqslant [q] \\ \theta_{[q]+1} = \{b, b+1, \ldots, 0, q-[q]\}, & b = \max\{-[q], -[\frac{m}{q-1}]\}. \end{cases}$$

Since $\theta = \theta_1$, the infinite word $\omega = \lim_{n\to\infty} \sigma^n(\theta)$ is one fix point of the map σ.

Define a map $h : \mathscr{A} \to \mathbb{R}$ by $\xi \in \mathscr{A} \mapsto \|\xi\|$, where $\|\cdot\|$ is defined as in (2.3).

For $n \geqslant 0$, let $I_1, I_2, \ldots, I_{t(n)}$ denote all the nth net intervals contained in the interval $[0, q^n]$ in increasing order, and $\xi_1, \xi_2, \ldots, \xi_{t(n)}$ denote the corresponding nth colors of $I_1, I_2 \ldots, I_{t(n)}$. Then by our construction, the word $\xi_1\xi_2 \ldots \xi_{t(n)}$ is just $\sigma^n(\theta)$. Since the word $\sigma^n(\theta)$ is always a prefix of $\sigma^{n+m}(\theta)$ for each $m \geqslant 1$, it follows that for any integer $m0$, $\Xi_{n+m} \cap [0, q^n] = \Xi_n \cap [0, q^n]$. Denoted by k_n the greatest number k such that $y_k \in [0, q^n)$, then

$\{y_{k+1} - y_k\}_{0 \leqslant k \leqslant k_n}$ is just the image of the word $\sigma^n(\theta)$ under the map h. This proves Theorem 2.1. $\square$

Remark 2.4. The substitution σ we constructed in the above proof is not of constant length whenever q is not an integer. To see this, we note that $|\sigma(\theta)| = [q] + 1 > q$; and $|\sigma^n(\theta)|$ is equal to the number of nth net intervals contained in $[0, q^n]$, which implies that $C^{-1}q^n < |\sigma^n(\theta)| < Cq^n$ for some constant $C > 0$. Thus $|\sigma^n(\theta)| \neq ([q] + 1)^n$ for some $n > 0$, which implies that σ is not of constant length. However, we do not know in general whether $\{y_{k+1} - y_k\}_{0 \leqslant k \leqslant k_n}$ can be an image of a substitution of constant length.

Example 2.5. Take $q = (\sqrt{5} + 1)/2$ and $m = 1$. In this case, by a simple calculation, the color set is $\mathscr{A} = \{a, b, c, d\}$, where

$$a = \{0, 1\}, \quad b = \{-1, 0, q - 1\}, \quad c = \{1 - q, 0, 1\}, \quad d = \{-1, 0, 1\}$$

and

$$a \to ab, \quad b \to c, \quad c \to db, \quad d \to cb.$$

The above substitution can be reduced to the substitution $\sigma(a) = ab$, $\sigma(b) = a$, which is usually called the *Substitution of Fibonacci*. The infinite sequence

$$\omega = \lim_{n \to \infty} \sigma^n(a) = abaababaabaababaababa\cdots$$

is the unique fixed point of σ on $A^{\mathbb{N}}$. Define $h : \mathscr{A} \to \mathbb{R}$ by $h(a) = 1$ and $h(b) = q - 1$. Then the difference sequence $\{y_{k+1} - y_k\}_{k \geqslant 0}$ is just the image of ω under the map h.

Furthermore, let $m = 1$, let $\ell \geqslant 3$ be an integer and $q = q_\ell > 1$ the positive root of the polynomial $q^\ell - q^{\ell-1} - \cdots - q - 1$. In this case, by some calculations, we get a substitution as follows: the color set $\mathscr{A} = \{1, 2, \ldots, \ell\}$, the substitution rule σ is defined by

$$1 \mapsto 12, \; 2 \mapsto 13, \; \ldots, \; (\ell - 1) \mapsto 1\ell, \; \ell \mapsto 1,$$

this substitution is called the *Rauzy Substitution over l letters*.

Define h by $h(1) = 1$ and $h(i) = q^{i-1} - \sum_{j=0}^{i-2} q^j$ for $2 \leqslant i \leqslant \ell$. The difference sequence $\{y_{k+1} - y_k\}_{k \geqslant 0}$ is just the image of the substitution sequence $\lim_{n \to \infty} \sigma^n(1)$ under the map h.

Remark 2.6. It was pointed out by the referee that Bugeaud had obtained the results in the above example in his dissertation [B1, Théorème 4, p. 130], by using a different method.

3. An algorithm to determine $\inf_k (y_{k+1} - y_k)$

Before giving our algorithm, we would like to recall some known results about the determination of $\inf_k (y_{k+1} - y_k)$.

Theorem 3.1 (Erdös et al. [EJJ]). *Given an integer $r > 1$, let q be the unique positive root of the polynomial $q^r - q^{r-1} - \cdots - q - 1$. Let $m = 1$. Then* $\inf_k (y_{k+1} - y_k) = 1/q$.

Theorem 3.2 (Komornik et al. [KLP]). (i) *Let $q \approx 1.466$ be the root of the polynomial $q^3 - q^2 - 1$ and $m = 1$, then* $\inf_k (y_{k+1} - y_k) = q^2 - 1$.

(ii) *Let $q = (\sqrt{5} + 1)/2$ be the golden ratio. Fix a positive integer m. Let ℓ be the integer defined by $q^{\ell-2} < m \leqslant q^{\ell-1}$, then*

$$\inf_k (y_{k+1} - y_k) = |F_\ell q - F_{\ell+1}|,$$

where $\{F_i\}_{i \geqslant 0}$ is the Fibonacci sequence $0, 1, 1, 2, 3, \ldots$, which satisfies the recurrence relation $F_i = F_{i-1} + F_{i-2}$ with the initial condition $F_0 = 0$ and $F_1 = 1$.

The proof of Theorem 2.1 contains an algorithm to determine all the possible distinct values of $(y_{k+1} - y_k)$ whenever q is a Pisot number and $m \geqslant q - 1$.

To see this, let σ be the substitution over $\mathscr{A}$ introduced as in the above proof and θ be the color of the 1st net interval $[0, 1]$. An element ξ in $\mathscr{A}$ is said to be *relative* to θ if there exists n such that ξ is a letter in the word $\sigma^n(\theta)$. Since we have had an easy algorithm to determine the word $\sigma(\eta)$ for $\eta \in \mathscr{A}$, we can determine the set $\mathscr{B}$ of all the elements relative to θ in a finite number of steps. The set of all the possible distinct values of $(y_{k+1} - y_k)$ is nothing but the set $\{\|\xi\|\colon \xi \in \mathscr{B}\}$.

However, the above algorithm consists of a large amount of computations when q is of high degree. In this section, we will give another much faster algorithm. Set for each integer $n \geqslant 0$,

$$E_n = \left\{ \sum_{i=0}^{n} \varepsilon_i q^i\colon\ \varepsilon_i \in \{-m, -m+1, \ldots, 0, \ldots, m\} \right\}$$

and $\Lambda_n = E_n \cap [0, m/(q-1)]$. Define $E = \bigcup_{n \geqslant 0} E_n$ and $\Lambda = \bigcup_{n \geqslant 0} \Lambda_n$. Our algorithm is based on the following simple lemma:

Lemma 3.1. *If q is a Pisot number and $m \geqslant q - 1$, then*

(i) *$\Lambda_n \subset \Lambda_{n+1}$ for every integer $n \geqslant 0$;*
(ii) $\inf_k (y_{k+1} - y_k) = \min\{z \neq 0\colon z \in \Lambda\}$;
(iii) *if $\Lambda_{n_0+1} = \Lambda_{n_0}$ for some integer n_0, then $\Lambda = \Lambda_{n_0}$.*

Proof. Statement (i) is trivial, it suffices to prove statements (ii) and (iii).

The inequality "$\geqslant$" in statement (ii) is clear, since $y_{k+1} - y_k \in \Lambda$ for each $k \geqslant 0$ by Lemma 1.1. To see the opposite inequality "$\leqslant$", note that every $z \in \Lambda_n$ with $z \neq 0$ can be written as $\sum_{i=0}^{n} \varepsilon_i q^i - \sum_{i=0}^{n} \varepsilon'_i q^i$, where

$\varepsilon_i, \varepsilon_i' \in \{0, 1, \ldots, m\}$ for $0 \leqslant i \leqslant n$. That is, $z = y_s - y_t$ for some integers $s > t \geqslant 0$, thus $z \geqslant \inf_k (y_{k+1} - y_k)$.

To prove statement (iii), it suffices to establish the following relation:

$$\Lambda_{n+1} = \left(\bigcup_{i=-m}^{m} (\pm q\Lambda_n + i) \right) \cap \left[0, \frac{m}{q-1}\right], \quad \forall n \geqslant 0. \tag{3.1}$$

To see the above equality, pick any $z \in \Lambda_{n+1}$ and suppose $z = \sum_{i=0}^{n+1} \varepsilon_i q^i$ where $\varepsilon_i \in \{-m, -m+1, \ldots, m\}$ for $0 \leqslant i \leqslant n+1$. Since $z \in [0, m/(q-1)]$,

$$\frac{z - \varepsilon_0}{q} = \sum_{i=0}^{n} \varepsilon_{i+1} q^i \in E_n$$

and

$$|(z - \varepsilon_0)/q| \leqslant (z + m)/q \leqslant (m/(q-1) + m)/q = m/(q-1).$$

That means $|(z - \varepsilon_0)/q| \in \Lambda_n$, therefore "$\subset$" holds in (3.1). One can check the relation "$\supset$" easily. Thus we complete the proof of statement (iii). □

By the above lemma, to determine $\inf_k (y_{k+1} - y_k)$, it suffices to determine the set Λ. In the following, we give a recursive algorithm to determine Λ.

For each real number z, denote by $T(z)$ the set

$$\{\pm qz + i \colon i \in \{-m, -m+1, \ldots, m\}\} \cap [0, m/(q-1)].$$

Note that $\Lambda_0 = \{0, 1, \ldots, \ell\}$ with $\ell = \min\{m, [m/(q-1)]\}$, here $[m/(q-1)]$ denotes the integral part of $m/(q-1)$. By (3.1), we can determine Λ_1 by $\Lambda_1 = \bigcup_{z \in \Lambda_0} T(z)$.

Suppose we have determined the set Λ_n with $\Lambda_n \neq \Lambda_{n-1}$, then we obtain the set Λ_{n+1} by

$$\Lambda_{n+1} = \Lambda_n \cup \left(\bigcup_{z \in \Lambda_n \setminus \Lambda_{n-1}} T(z) \right).$$

Since as in the proof of Theorem 2.1, the cardinality of Λ does not exceed

$$N_2 = \frac{m}{q-1} \frac{(2m)^{d-1}}{\prod_{i=1}^{d-1} (1 - |\alpha_i|)},$$

where d is the degree of q and $\alpha_1, \ldots, \alpha_{d-1}$ the algebraic conjugates of q, there exists $n_0 \leqslant N_2$ such that $\Lambda_{n_0} = \Lambda_{n_0 - 1}$ and thus $\Lambda = \Lambda_{n_0}$.

Example 3.2. Given $r \in \mathbb{N}$, $m = 1$. Let q denote the unique positive real solution of the equation

$$q^r = q^{r-1} + q^{r-2} + \cdots + 1.$$

This implies

$$\frac{1}{q-1} = 1 + \frac{1}{q^r - 1} \quad \text{and} \quad \frac{1}{q} = q^{r-1} - q^{r-2} \cdots - 1.$$

By a simple calculation, we get

$$\Lambda_0 = \{0, 1\}, \ \ T(1) = \{q-1\}, \ \ T^2(1) = T(q-1) = \{q^2-q-1\}, \ldots$$

and

$$T^{r-1}(1) = T(q^{r-2} - q^{r-3} - \cdots - 1) = \left\{ q^{r-1} - q^{r-2} - \cdots - 1 = \frac{1}{q} \right\}.$$

Notice that $T^r(1) = T(\frac{1}{q}) = \{0, 1\}$, $\Lambda_r = \Lambda_{r-1}$, we get therefore by Theorem 3.3(iii), $\Lambda = \Lambda_{r-1}$, and

$$\inf_{k \geqslant 0} (y_{k+1} - y_k) = \inf(\Lambda \backslash \{0\}) = 1/q,$$

this is exactly the conclusion of Theorem 3.1.

Using the above algorithm, by computation experiments we can easily determine $\inf_k (y_{k+1} - y_k)$ for Pisot numbers q of small degree and small integer m. In the following Tables 1–3 we give some results by computation experiments.

Table 1
$\inf_k (y_{k+1} - y_k)$ corresponding to the first nine smallest Pisot number q and $m = 1$

Polynomial for q	Numer. est. of q	$\inf_k (y_{k+1} - y_k)$	Numer. est. of $\inf_k (y_{k+1} - y_k)$
$x^3 - x - 1$	1.324717	$-3q^2 + q + 4$	0.06008495
$x^4 - x^3 - 1$	1.380277	$q^3 - 4q^2 + 5$	0.00899345
$x^5 - x^4 - x^3 + x^2 - 1$	1.443268	$4q^2 - 3q - 4$	0.00229284
$x^3 - x^2 - 1$	1.465571	$q^2 - 2$	0.14789903
$x^6 - x^5 - x^4 + x^2 - 1$	1.501594	$-q^5 - 2q^4 + 4q^3 + 3q^2 - 3q + 2$	0.00034913
$x^5 - x^3 - x^2 - x - 1$	1.534157	$-2q^4 + 3q^3 - q^2 + 3q - 2$	0.00215591
$x^7 - x^6 - x^5 + x^2 - 1$	1.545215	$-3q^6 + 2q^5 + 8q^4 - 2q^3 - 8q^2 + 2q + 1$	0.00004243
$x^6 - 2x^5 + x^4 - x^2 + x - 1$	1.561752	$-5q^5 + 8q^4 + q^3 - 3q^2 + 6q - 7$	0.00022195
$x^5 - x^4 - x^2 - 1$	1.570147	$q^4 - 2q^2 - 2q + 2$	0.00699287

Table 2
$q \approx 1.46557123$ be the real root of $x^3 - x^2 - 1 = 0$ and $1 \leqslant m \leqslant 10$

m	$\inf_k (y_{k+1} - y_k)$	Numer. est. of $\inf_k (y_{k+1} - y_k)$
1	$q^2 - 2$	0.14789903
2	$-3q^2 + q + 5$	0.02187412
3	$q^2 + 4q - 8$	0.01018396
4	$-8q^2 + 9q + 4$	0.00694880
5	$9q^2 - 5q - 12$	0.00323516
6	$9q^2 - 5q - 12$	0.00323516
7	$-5q^2 - 7q + 21$	0.00150619
8	$-5q^2 - 7q + 21$	0.00150619
9	$21q^2 - 26q - 7$	0.00102772
10	$21q^2 - 26q - 7$	0.00102772

Table 3
$q \approx 1.32471795$ be the real root of $x^3 - x - 1 = 0$ and $1 \leqslant m \leqslant 10$

m	$\inf_k (y_{k+1} - y_k)$	Numer. est. of $\inf_k (y_{k+1} - y_k)$
1	$-3q^2 + q + 4$	0.06008495
2	$-7q^2 + 4q + 7$	0.1472816
3	$-4q^2 - 3q + 11$	0.00633546
4	$10q^2 - 14q + 1$	0.00272526
5	$q^2 + 10q - 15$	0.00205723
6	$-15q^2 + q + 25$	0.00155296
7	$25q^2 - 15q - 24$	0.00117229
8	$9q^2 - 24q + 16$	0.00066802
9	$9q^2 - 24q + 16$	0.00066802
10	$16q^2 + 9q - 40$	0.00050427

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References

[Al] J.-P. Allouche, Automates finis en théorie des nombre, Exposition. Math. 5 (3) (1987) 239–266.

[BF] D. Berend, C. Frougny, Computability by finite automata and Pisot bases, Math. Systems Theory 27 (1994) 275–282.

[BDGPS] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J.-P. Schreiber, Pisot and Salem Numbers, Birkhäuser-Verlag, Basel, 1992.

[B] Y. Bugeaud, On a property of Pisot numbers and related questions, Acta Math. Hungar. 73 (1996) 33–39.

[B1] Y. Bugeaud, Formes linéaires de logarithmes et applications, Dissertation, Université Strasbourg I, Strasbourg, 1996, Prépublication de l'Institut de Recherche Mathématique Advancée, 1996/7.

[EJJ] P. Erdös, M. Joó, I. Joó, On a problem of Tamás Varga, Bull. Soc. Math. France 120 (1992) 507–521.

[EJK1] P. Erdös, I. Joó, V. Komornik, Characterization of the unique expansions $1 = \sum q^{-n_i}$ and related problems, Bull. Soc. Math. France 118 (1990) 377–390.

[EJK2] P. Erdös, I. Joó, V. Komornik, On the sequence of numbers of the form $\varepsilon_0 + \varepsilon_1 q + \cdots + \varepsilon_n q^n$, $\varepsilon_i \in \{0, 1\}$, Acta Arith. 83 (3) (1998) 201–210.

[EK] P. Erdös, V. Komornik, On developments in non-integer bases, Acta Math. Hungar. 79 (1–2) (1998) 57–83.

[F] C. Frougny, Representations of number and finite automata, Math. Systems Theory 25 (1992) 37–60.

[G] A.M. Garsia, Arithmetic properties of Bernoulli convolutions, Trans. Amer. Math. Soc. 102 (1962) 409–432.

[KLP] V. Komornik, P. Loreti, M. Pedicini, An approximation property of Pisot numbers, J. Number Theory 80 (2000) 218–237.

[Q] M. Queffélec, Substitution Dynamical Systems-Spectral Analysis, in: Lecture Notes in Mathematics, Vol. 1294, Springer-Verlag, Berlin, New York, 1987.

[Sa] R. Salem, Algebraic Numbers and Fourier Analysis, Heath Mathematical Monographs, Heath, Boston, 1963.

[Sh] J. Shallit, Number theory and formal languages. Emerging applications of number theory (Minneapolis, MN, 1996), 547–570, IMA Vol. Math. Appl., 109, Springer, New York, 1999.