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Ergodic Limits on the Conformal Repellers[1]

De-Jun Feng

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China and The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Hong Kong
E-mail: dfeng@math.tsinghua.edu.cn

Ka-Sing Lau[2]

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong
E-mail: kslau@math.cuhk.edu.hk

and

Jun Wu

Department of Mathematics and Nonlinear Science Center, Wuhan University, Wuhan, 430072, People's Republic of China and The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Hong Kong
E-mail: wujunyu@public.wh.hb.cn

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Let J be the repeller of an expanding, $C^{1+\delta}$-conformal topological mixing map g. Let $\Phi : J \to \mathbb{R}^d$ be a continuous function and let $\alpha(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\Phi(g^j x)$ (when the limit exists) be the ergodic limit. It is known that the possible $\alpha(x)$ are just the values $\int \Phi\, d\mu$ for all g-invariant measures μ. For any α in the range of the ergodic limits, we prove the following variational formula:

$$\dim\{x \in J : \alpha(x) = \alpha\} = \max_{\mu}\left\{\frac{h_g(\mu)}{\int \log \|D_x g\| d\mu(x)} : \int \Phi\, d\mu = \alpha\right\},$$

where μ is a g-invariant Borel probability measure on J, $h_g(\mu)$ is the entropy of μ, $\|D_x g\|$ is the operator norm of the differential $D_x g$, and dim is the Hausdorff dimension or the packing dimension. This result gives a substantial extension of the well-known case that Φ is Hölder continuous. We also prove that unless the same

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[2]To whom correspondence should be addressed.

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ergodic limit exists everywhere, the set of points whose ergodic limit does not exist has the same Hausdorff dimension as the whole space J.

Key Words: conformal repeller; dimensions; entropies; recurrences; Markov measures.

1. INTRODUCTION

Let M be a smooth Riemannian manifold and $g: M \to M$ a $C^{1+\delta}$-conformal map. Consider a g-invariant compact subset J of M. We say that g is *expanding* on J if there exist $C > 1$ such that $\|(D_x g)u\| \geqslant C\|u\|$ for all $x \in J$, $u \in T_x M$. We say that J is a *repeller* of the expanding g if:

(a) $J = \bigcap_{n \geqslant 0} g^{-n} V$ for some open neighborhood V of J; and
(b) g is topologically mixing on J, i.e., when U, W are nonempty (relative) open sets in J, $g^n(U) \cap W \neq \emptyset$ for n sufficiently large.

A finite closed cover $\{R_0, \ldots, R_{m-1}\}$ of J is called a *Markov partition* of J (with respect to g) if:

(i) $\overline{\operatorname{int} R_i} = R_i$ for each $i = 0, \ldots, m-1$;
(ii) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ for $i \neq j$; and
(iii) each $g(R_i)$ is the union of a subfamily of $\{R_j\}_{j=0}^{m-1}$.

It is well known that any repeller J of a continuously differentiable expanding map g has Markov partition of arbitrary small diameter (see [25, p. 146]) and (J, g) is semi-conjugated to (Σ_A, σ), a subshift space of finite type. In what follows, we always assume that J is a repeller of an expanding, $C^{1+\delta}$-conformal, topologically mixing map g.

Let Φ be a continuous function defined on J with values in $\mathbb{R}^d$. For any $x \in J$, we define the ergodic limit, when it exists, as

$$\alpha(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(g^j x).$$

The quantity $\alpha(x)$ is regarded as the *recurrence* of x relative to Φ (the term "recurrence" gets its usual sense when $\Phi = (1_{B_1}, \ldots, 1_{B_d})$ where 1_B denotes the characteristic function of a set B). Let

$$L_\Phi = \{\alpha : \alpha = \alpha(x) \quad \text{for some } x \in J\}.$$

Let $\mathcal{M}_g(J)$ be the set of all g-invariant Borel probability measures concentrated on J. The function $\Phi : J \to \mathbb{R}^d$ induces a map $\Phi_* : \mathcal{M}_g(J) \to \mathbb{R}^d$ given by

$$\Phi_*(\mu) = \int_J \Phi \, d\mu, \qquad \mu \in \mathcal{M}_g(J).$$

As a consequence of the Birkhoff ergodic theorem, we see that if μ is ergodic, then $\alpha(x) = \Phi_*(\mu)$ for μ-a.a. $x \in J$. It can be proved that $L_\Phi = \Phi_*(\mathscr{M}_g(J))$ and hence L_Φ is a nonempty compact convex set. For $\alpha \in L_\Phi$, we let

$$E(\alpha) = \{x \in J : \alpha(x) = \alpha\}$$

and

$$\mathscr{F}_\Phi(\alpha) = \{\mu \in \mathscr{M}_g(J) : \ \Phi_*(\mu) = \alpha\}.$$

In this paper, we will investigate the size of the set $E(\alpha)$ as well as the size of the set of points such that the limits defining $\alpha(x)$ do not exist. Recall that J is a metric space induced by the Riemannian metric and there are various notions of dimension on J. We will consider the Hausdorff dimension $\dim_{\mathrm{H}}$ and the packing dimension $\dim_{\mathrm{P}}$ (see, e.g., [11, 12, 19, 21]). The sizes of the sets in question will be described by these dimensions.

The first historical example of this type is due to Besicovitch [1] and Eggleston [10]; they proved that for $\alpha \in I = [0, 1]$,

$$E(\alpha) = \left\{ x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n} \in I : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j = \alpha, \ \varepsilon_n = 0 \text{ or } 1 \right\}$$

has Hausdorff dimension $-(\alpha \log_2 \alpha + (1-\alpha)\log_2(1-\alpha))$. In this case the corresponding maps are $g : I \to I$ such that $g(x) = 2x \pmod 1$ and $\Phi(x) = \chi_{[\frac{1}{2},1]}$ (they are not continuous). A slightly more elaborate example was given by Billingsley [2]. Fan and Lau [15] studied the asymptotic behavior at infinity of multiperiodic functions $F(x) = \prod_{n=1}^{\infty} f(\frac{x}{2^n})$ where f is a positive Hölder continuous periodic function with period 1 (e.g., $F(x) = |\hat{\phi}(x)|^q$ where $\hat{\phi}(x)$ is the Fourier transform of the scaling functions in the wavelet theory). By using the Ruelle–Perron–Frobenius operator with the Hölder continuous potential $\log f$ and the standard "multifractal formalism" argument, they showed that ($\Phi = \log f$, and $g(x) = 2x \pmod 1$ as above) the Hausdorff dimension of

$$E(\alpha) = \left\{ x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(2^j x) = \alpha \right\}$$

is $h_g(\mu)/\log 2$ where $h_g(\mu)$ is the entropy of the Gibbs measure μ with respect to g [15, Theorem 6]. Some further consideration of the ergodic limit and the multifractal formalism for Hölder continuous Φ was given by Pesin and Weiss [23]; a special case $\Phi(x) = \|D_x g\|$ was considered by Weiss [27].

If the function Φ has no regularity like Hölder continuity or summable variation, the question is more subtle because the multifractal formalism will not work as there is a lack of differentiability on the pressure function and a lack of Gibbs property on the invariant measures. Fan *et al.* [14] had considered the setting on (Σ_A, σ), a subshift space of finite type. They showed that for a continuous $\Phi : \Sigma_A \to \mathbb{R}^d$,

$$\dim_{\mathrm{H}} E(\alpha) = \frac{1}{\log m} \max\{h_\sigma(\mu) : \mu \in \mathscr{F}_\Phi(\alpha)\}.$$

Another consideration using pressures was given by Olivier [20]. Our main results here are the following.

THEOREM 1.1. *Let J be a repeller of an expanding, $C^{1+\delta}$-conformal topological mixing map g. Let $\Phi : J \to \mathbb{R}^d$ be a continuous function. Then for any $\alpha \in L_\Phi$, we have the variational formula*

$$\dim_{\mathrm{H}} E(\alpha) = \dim_{\mathrm{P}} E(\alpha) = \max_{\mu \in \mathscr{F}_\Phi(\alpha)} \frac{h_g(\mu)}{\int \log \|D_x g\| \, d\mu(x)},$$

where $\|D_x g\|$ is the operator norm of the differential $D_x g$, $h_g(\mu)$ is the entropy of μ with respect to g. Moreover, $\dim_{\mathrm{H}} E(\alpha)$ *is an upper semi-continuous function of α.*

THEOREM 1.2. *Under the hypotheses of Theorem* 1.1, *either*

(i) *all points $x \in J$ have the same ergodic limit*; *or*

(ii) *the set of points x such that the limit defining* $\alpha(x)$ *does not exist is of the same Hausdorff dimension as that of J.*

A first thought of proving these theorems is to lift the dynamical system (J, g) to (Σ_A, σ) and apply the results in [14]. However, this will meet some difficulties. Firstly, in contrast to the Hölder continuous case, it is possible that for some $\alpha \in L_\Phi$, there exists no ergodic measure μ supported on $E(\alpha)$; we cannot compare the dimensions of $E(\alpha)$ and its lift by using the measure μ in the usual way (see, e.g., [16, pp. 341,342]). Secondly, the lifting map is not one-to-one on the boundary of the Markov partition and we cannot make use of the ergodic measure to take care of the boundary (by ignoring a measure zero set). Hence instead of using directly the results in the symbolic space, we will adopt the approach by modifying the well-known topological pressure and Bowen's formula (see [21]). We introduce the following expressions (Section 4): For each real number s, we define

$$f(\alpha, s; n, \varepsilon) = \sum_{[\omega] \in F(\alpha; n, \varepsilon)} \operatorname{diam}(R_\omega)^s$$

with

$$F(\alpha; n, \varepsilon) = \left\{ [\omega] : \omega \in \Sigma_{A,n},\ \left| \frac{1}{n} \sum_{j=0}^{n-1} \Phi(g^j x) - \alpha \right| < \varepsilon \quad \text{for some } x \in R_\omega \right\}.$$

Here $\Sigma_{A,n}$ is the n-tuple in Σ_A and $R_\omega = \pi([\omega])$ with $\pi : \Sigma_A \to J$ denoting the coding map. We prove (Proposition 4.3 and Theorem 5.1)

THEOREM 1.3. *For $\alpha \in L_\Phi$ and $s \in \mathbb{R}$, we have*

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{\log f(\alpha, s; n, \varepsilon)}{n} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log f(\alpha, s; n, \varepsilon)}{n} := P(\alpha, s).$$

Moreover $P(\alpha, s)$ satisfies the following variational principle:

$$P(\alpha, s) = \max\{h_g(\mu) - s \int \log \|D_x g\| \, d\mu(x) : \ \mu \in \mathscr{F}_\Phi(\alpha)\}.$$

For each $\alpha \in L_\Phi$, if we define $\Lambda(\alpha)$ such that $P(\alpha, \Lambda(\alpha)) = 0$, then we can prove Theorem 1.1 by showing that

$$\dim_{\mathrm{H}} E(\alpha) = \dim_{\mathrm{P}} E(\alpha) = \Lambda(\alpha) \tag{1.1}$$

and

$$\Lambda(\alpha) = \max_{\mu \in \mathscr{F}_\Phi(\alpha)} \frac{h_g(\mu)}{\int \log \|D_x g\| \, d\mu(x)}. \tag{1.2}$$

The invariant set considered in Theorem 1.2 is called the *divergence set* of Φ. The Birkhoff ergodic theorem says that the divergence set is of null measure with respect to any invariant measure. Theorem 1.2 states that it is either empty or large in the sense that it has the same Hausdorff dimension as that of the whole space. The result of full Hausdorff dimension for Hölder continuous Φ was obtained by Barreira and Schmeling [4] (see also [5, 7, 22]). In our proof we first make use of Bowen's formula (see, e.g., [21, p. 203]) to choose an α such that $\dim_{\mathrm{H}} J = \dim_{\mathrm{H}} E(\alpha)$, then use the symbolic expression to adjust the $x \in E(\alpha)$ to form a new set F of the same dimension, but each $y \in F$ does not have an ergodic limit.

We organize this paper as follows. In Section 2, we set up the necessary materials for the subshift space of finite type and the Markov measure of order ℓ. In Section 3, we prove a dimensional result for the "nonhomogeneous" Moran sets. This together with the mass distribution principle are used in Section 4 to prove (1.1). The variational principle of the pressure function $P(\alpha, s)$ in Theorem 1.3 is proved in Section 5 and the Markov measures of order ℓ, $\ell \geqslant 1$, are used to approximate the measure that attains the maximum in (1.2). By using the results in the previous sections, we complete the proofs of Theorems 1.1 and 1.2 in Section 6.

2. PRELIMINARIES

Most of the materials in this section are known. We recall them here both for our convenience and that of the reader.

For an $m \times m$ matrix A with entries 0 or 1, we let (Σ_A, σ) denote the subshift space of finite type [6]. Σ_A is a metric space with $d(x, y) = m^{-n}$ for $x = (x_j)_{j \geqslant 1}$ and $y = (y_j)_{j \geqslant 1}$, where n is the largest integer such that $x_j = y_j, 1 \leqslant j \leqslant n$. We shall always assume that A is primitive. This means the dynamical system (Σ_A, σ) is topologically mixing.

For $k \geqslant 1$, $\Sigma_{A,k}$ denotes the set of finite sequences $\omega = (i_1, \ldots, i_k)$ such that $a_{i_j, i_{j+1}} = 1$ for all $1 \leqslant j \leqslant k-1$. These sequences ω are called (*admissible*) *words*; the *length* of the word is denoted by $|\omega|$ $(= k)$. For $\omega = (i_1, \ldots, i_k) \in \Sigma_{A,k}$, the *k-cylinder* $[\omega]$ is defined by $\{x \in \Sigma_A : x_1 = i_1, \ldots, x_k = i_k\}$. It is clear that there is a one-to-one correspondence between $\Sigma_{A,k}$ and the set of k-cylinders. Without confusion we just use $\Sigma_{A,k}$ to denote the set of all k-cylinders.

Let ξ_n be the partition consisting of all n-cylinders $[\omega] \in \Sigma_{A,n}$. The entropy $h_\sigma(\mu)$ of a σ-invariant measure μ on Σ_A (i.e., $\mu \in \mathcal{M}_\sigma(\Sigma_A)$) can be expressed as

$$h(\mu) = \lim_{n \to \infty} \frac{H_{\xi_n}(\mu)}{n} \qquad \text{where} \quad H_{\xi_n}(\mu) = -\sum_{A \in \xi_n} \mu(A) \log \mu(A).$$

The *nth conditional entropy of* μ is defined by $h^{(n)}(\mu) = H_{\xi_n | \xi_{n-1}}(\mu), n > 1$. Using elementary properties of the conditional entropy [26, p. 80], we can prove

PROPOSITION 2.1. *For each* $\mu \in \mathcal{M}_\sigma(\Sigma_A)$, *we have*

$$h^{(n)}(\mu) = H_{\xi_n}(\mu) - H_{\xi_{n-1}}(\mu) \qquad \forall n > 1$$

and

$$h(\mu) = \lim_{n \to \infty} h^{(n)}(\mu) = \inf_n h^{(n)}(\mu).$$

The entropy $h(\mu)$ *is an upper semi-continuous functional defined on* $\mathcal{M}_\sigma(\Sigma_A)$ *with respect to the weak* topology.*

A Borel probability measure μ on Σ_A is uniquely determined by its values on the cylinders with

$$\sum_{\omega\in\Sigma_{A,n}} \mu([\omega]) = 1, \qquad \sum_i \mu([\omega, i]) = \mu([\omega]). \tag{2.1}$$

It is invariant with respect to the shift σ if and only if

$$\sum_i \mu([i,\omega]) = \mu([\omega]) \qquad \forall n \geqslant 1 \quad \text{and} \quad \omega \in \Sigma_{A,n}. \tag{2.2}$$

These three conditions may be referred to as normalization condition, consistence condition and invariance condition. We call a measure $\mu \in \mathscr{M}_\sigma(\Sigma_A)$ a *Markov measure of order ℓ* (or simply *ℓ-Markov measure*), $\ell \geqslant 1$, if it satisfies the following Markov property: for $n \geqslant 1$,

$$\mu([i_1, \ldots, i_{n+\ell}]) = \mu([i_1, \ldots, i_{n+\ell-1}]) \frac{\mu([i_n, \ldots, i_{n+\ell}])}{\mu([i_n, \ldots, i_{n+\ell-1}])}$$
$$\left(= \mu([i_1, \ldots, i_{\ell+1}]) \prod_{j=2}^{n} \frac{\mu([i_j, \ldots, i_{j+\ell}])}{\mu([i_j, \ldots, i_{j+\ell-1}])} \right).$$

Note that the standard Markov measure is when $\ell = 1$. We will see from Corollary 2.4 that the set of all ℓ-Markov measures, $\ell \geqslant 1$, is dense in $\mathscr{M}_\sigma(\Sigma_A)$.

Proposition 2.2. *Suppose $\mu \in \mathscr{M}_\sigma(\Sigma_A)$ is an ℓ-Markov measure. The entropy of μ is*

$$h(\mu) = -\sum_{i_1,\ldots,i_{\ell+1}} \mu([i_1, \ldots, i_{\ell+1}]) \log \frac{\mu([i_1, \ldots, i_{\ell+1}])}{\mu([i_1, \ldots, i_\ell])}.$$

Moreover, $h^{(n)}(\mu) = h(\mu)$ *for* $n \geqslant \ell + 1$.

Proof. We need only use the definition of Markov measure and the expression of $h^{(n)}(\mu)$ in Proposition 2.1 to check that $h^{(n)}(\mu) = h^{(n-1)}(\mu)$ for $n \geqslant \ell + 1$. ∎

In view of (2.1) and (2.2) we let Δ_k denote the set of all nonnegative functions p defined on $\Sigma_{A,k}$ satisfying the following two relations:

$$\sum_{i_1,i_2,\ldots,i_k} p(i_1, i_2, \ldots, i_k) = 1,$$

$$\sum_i p(i, i_1, \ldots, i_{k-1}) = \sum_i p(i_1, \ldots, i_{k-1}, i).$$

PROPOSITION 2.3. *Let $p \in \Delta_k$; define μ on Σ_A by*

$$\mu([i_1, \dots, i_n]) = p(i_1, \dots, i_k) \prod_{j=2}^{n-k+1} \frac{p(i_j, \dots, i_{j+k-1})}{\sum_i p(i_j, \dots, i_{j+k-2}, i)}.$$

Then μ is a $(k-1)$-Markov measure. If furthermore p is positive, then μ is the Gibbs measures associated with the potential

$$\psi(x) = \log p(x_1, \dots, x_k) - \log \sum_i p(x_1, \dots, x_{k-1}, i).$$

The reader can refer to [6] for the definition and property of a Gibbs measure. As a corollary, we have

COROLLARY 2.4. *Given $\mu \in \mathcal{M}_\sigma(\Sigma_A)$, let $\{p_k\}$ in Δ_k be defined by*

$$p_k(i_1, \dots, i_k) = \mu([i_1, \dots, i_k]).$$

Let μ_k be the associated $(k-1)$-Markov measure in Proposition 2.3. Then

$$\mu = w^*\text{-}\lim_{k\to\infty} \mu_k, \qquad h(\mu) = \lim_{k\to\infty} h(\mu_k).$$

If further the support of μ is the whole space Σ_A, then the $\{\mu_k\}$ are ergodic and are Gibbs measures.

We call μ_k the *kth Markov approximation* of μ.

COROLLARY 2.5. *For any $\mu \in \mathcal{M}_\sigma(\Sigma_A)$, there exists a sequence of ergodic measures $\{\nu_k\}_{k\geqslant 1} \subset \mathcal{M}_\sigma(\Sigma_A)$ such that*

$$\mu = w^*\text{-}\lim_{k\to\infty} \nu_k, \qquad h(\mu) = \lim_{k\to\infty} h(\nu_k).$$

Proof. In view of Corollary 2.4, we only need to consider the case that the support of μ is not the whole space Σ_A. In such a case we select a $\xi \in \mathcal{M}_\sigma(\Sigma_A)$ supporting the whole space Σ_A. Take $\mu^{(n)} = (1-\frac{1}{n})\mu + \frac{1}{n}\xi$. Then $\mu^{(n)}$ supports the whole space Σ_A. Since $\mu = w^*\text{-}\lim_{n\to\infty} \mu^{(n)}$ and $h(\mu) = \lim_{n\to\infty} h(\mu^{(n)})$, combining it with Corollary 2.4 we get the desired result. ∎

Let X be a compact metric space and let $T : X \to X$ be a continuous map. We let $\mathcal{M}_T(X)$ denote the space of all T-invariant Borel probability measures on X.

PROPOSITION 2.6. *Let $X_i, i = 1, 2$ be compact metric spaces and let $T_i : X_i \to X_i$ be continuous. Suppose $\pi : X_1 \to X_2$ is a continuous surjection such* *that the following diagram commutes*:

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ \pi\downarrow & & \downarrow\pi \\ X_2 & \xrightarrow{T_2} & X_2 \end{array}$$

Then $\pi^* : \mathcal{M}_{T_1}(X_1) \to \mathcal{M}_{T_2}(X_2)$ *(defined by* $\mu \mapsto \mu \circ \pi^{-1}$*) is surjective. If, furthermore, there is an integer* $m > 0$ *so that* $\pi^{-1}(y)$ *has at most m elements for each* $y \in X_2$, *then*

$$h_{T_1}(\mu) = h_{T_2}(\mu \circ \pi^{-1})$$

for each $\mu \in \mathcal{M}_{T_1}(X_1)$.

Proof. The first part of the result is the same as [18, Chap. IV, Lemma 8.3]. The second part follows from the Abramov–Rokhlin formula (see [3]). ∎

From Corollary 2.5 and Proposition 2.6, we have the following corollary immediately.

COROLLARY 2.7. *For any* $\mu \in \mathcal{M}_g(J)$, *there exists a sequence of ergodic measures* $\{\mu_k\}_{k\geqslant 1} \subset \mathcal{M}_g(J)$ *such that*

$$\mu = w^*\text{-}\lim_{k\to\infty} \mu_k, \quad h(\mu) = \lim_{k\to\infty} h(\mu_k).$$

3. NONHOMOGENEOUS MORAN SET

In our proof of (1.1) we need to use a class of Cantor sets from a very general Moran construction. Let $X \subset \mathbb{R}^d$ be a compact set with nonempty interior. Let $\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers. Let $D = \bigcup_{k\geqslant 0} D_k$ with $D_0 = \{\emptyset\}$ and $D_k = \{\omega = (j_1 j_2 \ldots j_k) : 1 \leqslant j_i \leqslant n_i, 1 \leqslant i \leqslant k\}$. Let $D_\infty = \{(j_1 j_2 \ldots) : 1 \leqslant j_i \leqslant n_i, i \geqslant 1\}$. Suppose that $\mathcal{G} = \{X_\omega : \omega \in D\}$ is a collection of subsets of $\mathbb{R}^d$. We say that $\mathcal{G}$ fulfills the *Moran structure* provided it satisfies the following conditions:

(1) $X_\emptyset = X$, $X_{\omega j} \subset X_\omega$ for any $\omega \in D_{k-1}, 1 \leqslant j \leqslant n_k$; the interiors of X_ω and $X_{\omega'}$ are disjoint $\forall \omega \neq \omega'$, $\omega, \omega' \in D_m$.

(2) There exist two positive constants C_1 and C_2, closed balls $\underline{B}_\omega$, $\bar{B}_\omega$ and $r_\omega \in \mathbb{R}^+$ for each $\omega \in D$, such that

(i) $\underline{B}_\omega \subset X_\omega \subset \bar{B}_\omega$;
(ii) $\underline{B}_\omega$ and $\bar{B}_\omega$ have radii $C_1 r_\omega$ and $C_2 r_\omega$;
(iii) $\lim_{k\to\infty} \max_{\omega\in D_k} r_\omega = 0$;
(iv) there exist positive constants C_3 and C_4 such that

$$C_3 \frac{r_{\omega\eta}}{r_\omega} \leqslant \frac{r_{\omega'\eta}}{r_{\omega'}} \leqslant C_4 \frac{r_{\omega\eta}}{r_\omega} \tag{3.1}$$

for all $\omega\eta \neq \omega'\eta,\ \omega, \omega' \in D_m,\ \omega\eta, \omega'\eta \in D_n, m \leqslant n.$

If $\mathscr{G}$ fulfills the above Moran structure, we call the set

$$E = \bigcap_{n>0} \bigcup_{\omega\in D_n} X_\omega$$

a *nonhomogeneous Moran set* associated with $\mathscr{G}$. The nonhomogeneity refers to the nonconstant number n_k of descendents in the kth level for each predecessor in the $(k-1)$th level. This class of sets was considered in [17, 24] for the case that each $X_\omega, \omega \in D_k$ has equal size. For the homogeneous Moran set (i.e., n_k is a constant) the reader can refer to [8, 9, 21] for details.

Let

$$\rho_k = \min_{(i_1\cdots i_k)\in D_k} \frac{r_{i_1\cdots i_k}}{r_{i_1\cdots i_{k-1}}}, \qquad M_k = \max_{(i_1\cdots i_k)\in D_k} r_{i_1\cdots i_k}.$$

They are the minimal contraction ratio and the maximal size (up to a fixed constant multiple) of the set X_ω in the kth generation.

PROPOSITION 3.1. *For the Moran set E defined as above, suppose furthermore*

$$\lim_{k\to\infty} \frac{\log \rho_k}{\log M_k} = 0. \tag{3.2}$$

Then we have

$$\dim_{\mathrm{H}} E = \liminf_{k\to\infty} s_k, \qquad \dim_{\mathrm{P}} E = \limsup_{k\to\infty} s_k,$$

where s_k satisfies the equation $\sum_{\omega\in D_k} r_\omega^{s_k} = 1$.

Proof. We first prove $\dim_{\mathrm{H}} E = \liminf_{k\to\infty} s_k$. The inequality $\dim_{\mathrm{H}} E \leqslant \liminf_{k\to\infty} s_k$ is straightforward. We need only prove the reverse inequality.

Let $\mathscr{B}_n$ be the σ-algebra generated by the cylinders $[\omega]$, $\omega \in D_n$. For any $n \geqslant 1$ and $\alpha \geqslant 0$, we define a measure μ_n (depends on α) on $\mathscr{B}_n$ by

$$\mu_n([\omega]) = \frac{r_\omega^\alpha}{\sum_{\omega' \in D_n} r_{\omega'}^\alpha} . \tag{3.3}$$

For any $m < n$ and for $\omega \in D_m$, we have

$$\begin{aligned}
\mu_n([\omega]) &= \sum_{\omega\eta \in D_n} \mu_n([\omega\eta]) \\
&= \left(\sum_{\omega\eta \in D_n} r_{\omega\eta}^\alpha \right) \Big/ \left(\sum_{\omega'\eta \in D_n} r_{\omega'\eta}^\alpha \right) \\
&= \sum_{\omega\eta \in D_n} \left((r_{\omega\eta}^\alpha r_\omega^{-\alpha}) r_\omega^\alpha \right) \Big/ \sum_{\omega'\eta \in D_n} \left((r_{\omega'\eta}^\alpha r_{\omega'}^{-\alpha}) r_{\omega'}^\alpha \right) \\
&\approx \mu_m([\omega]) \qquad \text{(by (3.1))}.
\end{aligned}$$

That is, there exists $C > 0$ such that for $m < n$,

$$C^{-1} < \frac{\mu_n([\omega])}{\mu_m([\omega])} < C, \qquad \forall \omega \in D_m.$$

By the compactness of D_∞, there is a subsequence $\{\mu_{n_k}\}_{k \geqslant 1}$ that converges in the weak* topology. Denote by μ the limit. Then

$$C^{-1} \leqslant \frac{\mu([\omega])}{\mu_m([\omega])} \leqslant C \qquad \forall m > 0 \quad \text{and} \quad \omega \in D_m. \tag{3.4}$$

Let ν be the probability measure on E such that $\nu = \mu \circ \pi^{-1}$ where $\pi : D_\infty \to E$ is defined by $(i_1 i_2 \cdots) \to \bigcap_{n>0} X_{i_1 \cdots i_n}$. We claim that for each $\alpha < \liminf_{k \to \infty} s_k$, there exists $C' > 0$ such that

$$\nu(X_\omega) \leqslant C' r_\omega^\alpha \qquad \forall \omega \in D_k \text{ with large } k.$$

To prove the claim, we observe that for any $\alpha < \beta < \liminf_{k \to \infty} s_k$, there exists N_0 such that for any $k \geqslant N_0$,

$$\sum_{\eta \in D_k} r_\eta^\beta > 1. \tag{3.5}$$

For $\omega \in D_n$, let

$$A = \{\eta \in D : r_\eta < r_\omega \leqslant r_{\eta^*},\ X_\eta \cap X_\omega \neq \emptyset\},$$

where $\eta = (j_1 \ldots j_m) \in D$ and $\eta^* = (j_1 \cdots j_{m-1})$. By Assumption 2(ii) of the Moran construction and a simple geometric argument, there exists

C_5 such that

$$\sum_{\eta\in A}\rho_{|\eta|}^d \leqslant \sum_{\eta\in A}\frac{r_\eta^d}{r_{\eta^*}^d} \leqslant \sum_{\eta\in A}\frac{r_\eta^d}{r_\omega^d} \leqslant C_5. \tag{3.6}$$

Hence for n large enough,

$$\begin{aligned}
\nu(X_\omega) &\leqslant \sum_{\eta\in A}\mu([\eta]) \\
&\leqslant C\sum_{\eta\in A}\frac{r_\eta^\alpha}{\sum_{(j_1\ldots j_{|\eta|})\in D_{|\eta|}} r_{j_1\ldots j_{|\eta|}}^\alpha} \qquad \text{(by (3.4))} \\
&\leqslant C\sum_{\eta\in A}\frac{r_\eta^\alpha M_{|\eta|}^{\beta-\alpha}}{\sum_{(j_1\ldots j_{|\eta|})\in D_{|\eta|}} r_{j_1\ldots j_{|\eta|}}^\beta} \qquad \text{(by (3.2))} \\
&\leqslant Cr_\omega^\alpha \sum_{\eta\in A} M_{|\eta|}^{\beta-\alpha} \qquad \text{(by (3.5))} \\
&\leqslant Cr_\omega^\alpha \sum_{\eta\in A} \rho_{|\eta|}^d \qquad \text{(by (3.2))} \\
&\leqslant CC_5 r_\omega^\alpha \qquad \text{(by (3.6))}.
\end{aligned}$$

This completes the proof of the claim.

Now let $\alpha' < \alpha$. By (3.2), there exists k_0 such that

$$r_{i_1\cdots i_{k-1}}^\alpha \leqslant r_{i_1\cdots i_k}^{\alpha'}, \qquad \forall k\geqslant k_0, \quad (i_1\cdots i_k)\in D_k. \tag{3.7}$$

Let B be a ball of radius r such that $r\leqslant \inf_{\omega\in D_{k_0+1}} r_\omega$. Set

$$\mathscr{B} = \left\{\omega\in D:\ X_\omega\cap B\neq\emptyset, \quad r_\omega\geqslant r > r_{\omega j},\ 1\leqslant j\leqslant n_{|\omega|+1}\right\}.$$

Then by assumption (2), there exists $L>0$ (not depending on r) such that $\#\mathscr{B}<L$. It follows from (3.7) that

$$\nu(B)\leqslant \sum_{\omega\in\mathscr{B}}\nu(X_\omega)\leqslant LCC_5 r_\omega^\alpha \leqslant LCC_5 r^{\alpha'}.$$

The mass distribution principle will imply that $\alpha'\leqslant \dim_{\mathrm H} E$. Since $\alpha'<\alpha$ is arbitrary, we have $\alpha\leqslant\dim_{\mathrm H} E$. Thus we obtain that $\dim_{\mathrm H} E\geqslant \liminf_{k\to\infty} s_k$.

We now prove that $\dim_{\mathrm P} E = \limsup_{k\to\infty} s_k$. Denote $s=\limsup_{k\to\infty} s_k$. We first show that $\dim_{\mathrm P} E\leqslant s$. Let $\varepsilon>0$; it suffices to show that $\dim_{\mathrm P} E\leqslant s+2\varepsilon$. Take $k_1\in\mathbb N$ such that $\frac{\log\rho_k}{\log M_k}\leqslant\frac{\varepsilon}{s+2\varepsilon}$ for any $k\geqslant k_1$. Let $\omega^* = (i_1\cdots i_{k-1})$ if $\omega=(i_1\cdots i_k)$; the definitions of ρ_k and M_k imply that

$$r_{\omega^*}\leqslant (r_\omega)^{(s+\varepsilon)/(s+2\varepsilon)} \qquad \forall k\geqslant k_1, \quad \omega\in D_k.$$

Take $k_2 > k_1$ such that $s_k < s + \varepsilon$ for each $k \geqslant k_2$. Let μ_n (depends on $s+\varepsilon$) be the measure defined as in (3.3) with $s+\varepsilon$ replacing α and let μ be a w^*-limit point of $\{\mu_n\}$. Then by (3.4), for each $n \geqslant k_2$ and $\omega \in D_n$,

$$\mu([\omega]) \geqslant C^{-1}\mu_n([\omega]) = C^{-1}\frac{r_\omega^{s+\varepsilon}}{\sum_{\omega' \in D_n} r_{\omega'}^{s+\varepsilon}} > C^{-1} r_\omega^{s+\varepsilon} \geqslant C^{-1} r_{\omega^*}^{s+2\varepsilon}.$$

Now pick $\delta > 0$ such that $\delta < \inf_{\omega \in D_{k_2}} |X_\omega|$ and suppose that $\{B_i\}$ is a family of disjoint balls with centers in E and diameters less than δ. For each i, pick $\omega_i \in D$ such that the center of B_i is contained in X_{ω_i}, and $X_{\omega_i} \subset B_i$ and $X_{\omega_i^*} \not\subset B_i$. It is clear that $\omega_i \in \bigcup_{k \geqslant k_2} D_k$ and $\sum_i \mu([\omega_i]) \leqslant 1$. Since $|X_{\omega_i^*}| > \frac{1}{2}|B_i|$ for each i, we have

$$\begin{aligned}\sum_i |B_i|^{s+2\varepsilon} &\leqslant \sum_i (2|X_{\omega_i^*}|)^{s+2\varepsilon} \leqslant (2C_2)^{s+2\varepsilon} \sum_i r_{\omega_i^*}^{s+2\varepsilon} \\ &\leqslant C(2C_2)^{s+2\varepsilon} \sum_i \mu([\omega_i]) \leqslant C(2C_2)^{s+2\varepsilon}.\end{aligned}$$

Hence by using the standard notations of packing dimension [11], we have

$$P_\delta^{s+2\varepsilon}(E) < C^{-1}(2C_2)^{s+2\varepsilon}$$

and $\mathscr{P}^{s+2\varepsilon}(E) \leqslant P_0^{s+2\varepsilon}(E) \leqslant P_\delta^{s+2\varepsilon}(E) < \infty$, which implies that $\dim_{\mathrm{P}} E \leqslant s + 2\varepsilon$.

To show that $\dim_{\mathrm{P}} E \geqslant s$, we consider the upper box dimension of $X_{\omega_0} \cap E$ for any fixed $k_0 \in \mathbb{N}$ and $\omega_0 \in D_{k_0}$. We need only show that $\overline{\dim}_{\mathrm{B}}(X_{\omega_0} \cap E) \geqslant s$. It follows that $\overline{\dim}_{\mathrm{B}}(V \cap E) \geqslant s$ for each open set V with $V \cap E \neq \emptyset$. This implies $\dim_{\mathrm{P}}(E) \geqslant s$ since E is compact [11].

Indeed for any $\varepsilon, \delta > 0$, take k such that $\max_{\omega_0\eta \in D_k} r_{\omega_0\eta} < \min\{\frac{1}{2}, \varepsilon\}$ and $s_k \geqslant s - \delta$. Then by (3.1),

$$\begin{aligned}1 &\leqslant \sum_{\omega' \in D_{k_0},\ \omega'\eta \in D_k} r_{\omega'\eta}^{s-\delta} \leqslant \sum_{\omega' \in D_{k_0},\ \omega'\eta \in D_k} \left(C_4 \frac{r_{\omega'}}{r_{\omega_0}} r_{\omega_0\eta}\right)^{s-\delta} \\ &\leqslant \left(\frac{C_4 \max_{\omega' \in D_{k_0}} r_{\omega'}}{r_{\omega_0}}\right)^{s-\delta} \# D_{k_0} \sum_{\omega_0\eta \in D_k} r_{\omega_0\eta}^{s-\delta} \\ &= C_6 \sum_{\omega_0\eta \in D_k} r_{\omega_0\eta}^{s-\delta}.\end{aligned}$$

Let $\mathscr{A}_n = \left\{\omega_0\eta \in D_k :\ 2^{-n-1} \leqslant r_{\omega_0\eta} < 2^{-n}\right\}$. Then

$$C_6^{-1} \leqslant \sum_{\omega_0\eta \in D_k} r_{\omega_0\eta}^{s-\delta} \leqslant \sum_{n=0}^{\infty} \# \mathscr{A}_n 2^{-n(s-\delta)},$$

which implies that there exists m such that

$$\#\mathscr{A}_m \geqslant C_6^{-1}\frac{1}{1-2^{-\delta}}2^{m(s-2\delta)}.$$

By conditions (1) and (2), there is a constant $C_7 \in \mathbb{N}$ (depending only upon C_1, C_2 and d) such that each closed ball with radius 2^{-m} intersects at most C_7 elements of $\{X_{\omega_0\eta} : \ \omega_0\eta \in \mathscr{A}_m\}$. Thus if we denote by $N_{2^{-m}}(X_{\omega_0} \cap E)$ the least number of closed balls with radii 2^{-m} needed to cover E, then

$$N_{2^{-m}}(X_{\omega_0} \cap E) \geqslant C_7^{-1}\#\mathscr{A}_m \geqslant C_6^{-1}C_7^{-1}\frac{1}{1-2^{\delta-s}}2^{-m(s-2\delta)}.$$

Since ε can be taken arbitrarily small, the number m in the above inequality can be picked arbitrarily large. Therefore $\overline{\dim}_{\mathrm{B}}(X_{\omega_0} \cap E) \geqslant s - 2\delta$. Since $\delta > 0$ is arbitrary, $\overline{\dim}_{\mathrm{B}}(X_{\omega_0} \cap E) \geqslant s$ for any $k_0 \in \mathbb{N}$ and $\omega_0 \in D_{k_0}$. This completes the proof of the claim and hence the proposition. ∎

4. PRESSURE FUNCTION AND DIMENSION OF $E(\alpha)$

Let $\mathscr{R} = \{R_0, \dots, R_{m-1}\}$ be a Markov partition of the repeller J with respect to an expanding, $C^{1+\delta}$-conformal, topological mixing map g. It is well known that this dynamical system induces a subshift space of finite type (Σ_A, σ), where $A = (a_{ij})$ is the transfer matrix of the Markov partition, namely, $a_{ij} = 1$ if $\operatorname{int} R_i \cap g^{-1}(\operatorname{int} R_j) \neq \emptyset$ and $a_{ij} = 0$ otherwise [25]. The matrix A is primitive, i.e., there is a positive integer M so that $A^M > 0$. This gives the coding map $\pi : \Sigma_A \to J$ such that

$$\pi(\omega) = \bigcap_{n \geqslant 1} g^{-(n-1)}(R_{i_n}), \qquad \forall \omega = (i_1 i_2 \cdots)$$

and the following diagram

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \pi\downarrow & & \downarrow\pi \\ J & \xrightarrow{g} & J \end{array}$$

commutes. The coding map π is a Hölder continuous surjection. Moreover, there is a positive integer q so that $\pi^{-1}(x)$ has at most q elements for each $x \in J$ (see [25, p. 147]). For each cylinder $[\omega]$, the set $\pi([\omega])$ is called a *basic set* and is denoted by R_ω. It follows from [21, Proposition 20.2] that

PROPOSITION 4.1.

(i) *For any integer* $k>0$, *the interiors of distinct* $R_\omega, \omega \in D_k$ *are disjoint.*

(ii) *Each* R_ω *contains a ball of radius* $\underline{r}_\omega$ *and is contained in a ball of radius* $\bar{r}_\omega$.

(iii) *There exist positive constants* $K_1 < 1$ *and* $K_2 > 1$ *such that for every* R_ω,

$$K_1 \prod_{j=0}^{n-1} \|D_{g^j(x)}g\|^{-1} \leqslant \underline{r}_\omega \leqslant \bar{r}_\omega \leqslant K_2 \prod_{j=0}^{n-1} \|D_{g^j(x)}g\|^{-1}, \qquad \forall x \in R_\omega. \tag{4.1}$$

Note that the second part makes use of $\|D_x g^n\| = \prod_{j=0}^{n-1} \|D_{g^j(x)}g\|$, a consequence of the chain rule and the property of the determinant.

Since $A^M > 0$, for any $\omega \in \Sigma_{A,n}$ and any $0 \leqslant z \leqslant m-1$, there are $0 \leqslant y_1, \ldots, y_M \leqslant m-1$ such that

$$(\omega, y_1, \ldots, y_M, z) \in \Sigma_{A,n+M+1}.$$

We call $\bar{\omega} = (\omega, y_1, \ldots, y_M)$ an *extension* of ω to join z.

For a fixed continuous $\Phi : J \to \mathbb{R}^d$, and for any $\alpha \in \mathbb{R}^d$, $n \geqslant 1$ and $\varepsilon > 0$, we define

$$F(\alpha; n, \varepsilon) = \left\{ [\omega] : \omega \in \Sigma_{A,n}, \ \left| \frac{1}{n} \sum_{j=0}^{n-1} \Phi(g^j x) - \alpha \right| < \varepsilon \quad \text{for some } x \in R_\omega \right\}$$

and

$$f(\alpha, s; n, \varepsilon) = \sum_{[\omega] \in F(\alpha; n, \varepsilon)} \operatorname{diam}(R_\omega)^s$$

for any $s \in \mathbb{R}$.

LEMMA 4.2. *For any* $\varepsilon > 0$, *there exists* $N = N(\varepsilon)$ *such that for any* $p \geqslant 1$ *if* $n \geqslant N$,

$$f(\alpha, s; n, \varepsilon)^p \leqslant C^{ps} f(\alpha, s; (n+M)p, 2\varepsilon)$$

for some $C > 0$, *independent of* p *and* ε.

Proof. Without loss of generality, we can assume that $|\Phi(x)| \leqslant 1$ for all $x \in J$ ($|\cdot|$ denotes the Euclidean norm).

For $\omega_1, \ldots, \omega_p \in F(\alpha; n, \varepsilon)$, let $\omega = \bar{\omega}_1 \cdots \bar{\omega}_p \in \sum_{A,p(n+M)}$ where $\bar{\omega}_k \in \Sigma_{A,n+M}$ is an extension of ω_k joining the leading letter of ω_{k+1} (with the convention that $\omega_{p+1} = \omega_1$). By definition we can choose $x_k \in R_{\omega_k}$, $1 \leqslant k \leqslant p$, such that

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\Phi(g^j x_k)-\alpha\right|<\varepsilon.$$

Let x be a point in R_ω; then $x=\pi(\bar{\omega}_1\cdots\bar{\omega}_p\cdots)$. Let $z_k=\pi(\bar{\omega}_k\cdots\bar{\omega}_p\cdots)$, $1\leqslant k\leqslant p$. It follows from the conjugation of g and σ that $g^{(n+M)(k-1)+j}(x)=g^j(z_k)$ and hence

$$\sum_{j=0}^{(n+M)p-1}\Phi(g^j x)=\sum_{k=1}^{p}\sum_{j=0}^{n+M-1}\Phi(g^j z_k).$$

Therefore,

$$\begin{aligned}
&\left|\frac{1}{(n+M)p}\sum_{j=0}^{(n+M)p-1}\Phi(g^j x)-\alpha\right|\\
&\quad=\left|\frac{1}{(n+M)p}\sum_{k=1}^{p}\sum_{j=0}^{n+M-1}\Phi(g^j z_k)-\alpha\right|\\
&\quad\leqslant\frac{1}{(n+M)p}\sum_{k=1}^{p}\left(\left|\sum_{j=0}^{n+M-1}(\Phi(g^j x_k)-\alpha)\right|+\sum_{j=0}^{n+M-1}\left|\Phi(g^j z_k)-\Phi(g^j x_k)\right|\right)\\
&\quad\leqslant\frac{1}{n+M}\left((n\varepsilon+2M)+\left(\sum_{j=0}^{n-1}\|\Phi\|_j+2M\right)\right),
\end{aligned}$$

where $\|\Phi\|_i=\sup\{|\Phi(x)-\Phi(y)|:\ x,y\in R_\omega \text{ for some } \omega\in\Sigma_{A,i}\}$. Since Φ is continuous, $n^{-1}\sum_{j=0}^{n-1}\|\Phi\|_j$ tends to zero as $n\to\infty$. We have

$$\left|\frac{1}{(n+M)p}\sum_{j=0}^{(n+M)p-1}\Phi(g^j x)-\alpha\right|\leqslant 2\varepsilon$$

for $n\geqslant N$ and for all $p\geqslant 1$. This implies that the $[\omega]$, which contains x, is in $F(\alpha;(n+M)p,2\varepsilon)$. Let $\lambda_{\max}=\max_{x\in J}\|D_x g\|$ and $\lambda_{\min}=\min_{x\in J}\|D_x g\|$. By (4.1), we have

$$\begin{aligned}
&\operatorname{diam}(R_{\bar{\omega}_1\bar{\omega}_2\cdots\bar{\omega}_p})\\
&\quad\geqslant K_1\prod_{j=0}^{(n+M)p-1}\|D_{g^j(x)}g\|^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= K_1 \left(\prod_{k=0}^{p-1} \prod_{j=0}^{n-1} \|D_{g^{(n+M)k+j}(x)} g\|^{-1} \right) \left(\prod_{k=0}^{p-1} \prod_{j=n}^{n+M-1} \|D_{g^{(n+M)k+j}(x)} g\|^{-1} \right) \\
&\geqslant K_1 \left(\prod_{k=0}^{p-1} \frac{1}{2K_2} \mathrm{diam}(R_{\omega_{k+1}}) \right) (\lambda_{\max})^{-Mp} \\
&\geqslant \left(\frac{K_1}{2K_2 \lambda_{\max}^M} \right)^p \prod_{k=1}^{p} \mathrm{diam}(R_{\omega_k}),
\end{aligned}
$$

and similarly

$$
\mathrm{diam}(R_{\bar{\omega}_1 \bar{\omega}_2 \cdots \bar{\omega}_p}) \leqslant \left(\frac{2K_2}{K_1 \lambda_{\min}^M} \right)^p \prod_{k=1}^{p} \mathrm{diam}(R_{\omega_k}).
$$

It follows that

$$
\begin{aligned}
\left(\sum_{[\omega] \in F(\alpha; n, \varepsilon)} \mathrm{diam}(R_\omega)^s \right)^p &\leqslant C^{ps} \sum_{[\omega_i] \in F(\alpha; n, \varepsilon)} \mathrm{diam}(R_{\bar{\omega}_1 \cdots \bar{\omega}_p})^s \\
&\leqslant C^{ps} \sum_{[\eta] \in F(\alpha; p(n+M), 2\varepsilon)} \mathrm{diam}(R_\eta)^s,
\end{aligned}
$$

where

$$
C = \begin{cases} K_1^{-1}(2K_2 \lambda_{\max}^M) & \text{for } s \geqslant 0 \\ (2K_2)^{-1}(K_1 \lambda_{\min}^M) & \text{for } s < 0 \end{cases}
$$

and the lemma follows. ∎

By using the above lemma, we have

PROPOSITION 4.3. *For $\alpha \in L_\Phi$ and $s \in \mathbb{R}$,*

$$
\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{\log f(\alpha, s; n, \varepsilon)}{n} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log f(\alpha, s; n, \varepsilon)}{n} =: P(\alpha, s). \qquad (4.2)
$$

The function $P(\alpha, s)$ is upper semi-continuous on the variable α. Moreover,

$$
C_2(t - s) \leqslant P(\alpha, s) - P(\alpha, t) \leqslant C_1(t - s) \qquad (4.3)
$$

for any $s, t \in \mathbb{R}$ with $s < t$, where $C_1 = \max_{x \in J} \log \|D_x g\|$ and $C_2 = \min_{x \in J} \log \|D_x g\|$.

Proof. The equality of the limits follows from the subadditivity in the lemma and a standard argument. We will include a proof for completeness. First, note that the two limits in (4.2) exist since $f(\alpha, s; n, \varepsilon)$ is an increasing function on the variable ε. Denote by β the left-hand-side limit. Then for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that

$$\liminf_{n\to\infty} \frac{\log f(\alpha, s; n, \varepsilon_0)}{n} < \beta + \delta.$$

Fix $\delta, \varepsilon_0 > 0$. To show the equality in (4.2), we only need to show that

$$\limsup_{n\to\infty} \frac{\log f(\alpha, s; n, \varepsilon_0/4)}{n} < \beta + 2\delta.$$

Fix $n \in \mathbb{N}$ with $n \geqslant N(\varepsilon_0/4)$ where $N(\varepsilon_0/4)$ is defined as in Lemma 4.2. Take a sequence of integers $n_k \uparrow \infty$ such that

$$f(\alpha, s; n_k, \varepsilon_0) < e^{n_k(\beta+\delta)}, \qquad \forall k \in \mathbb{N}.$$

For each k, write $n_k = (n+M)p_k - \ell_k$ with $0 \leqslant \ell_k < n+M$. By Lemma 4.2,

$$f(\alpha, s; n, \varepsilon_0/4)^{p_k} \leqslant C^{p_k s} f(\alpha, s; (n+M)p_k, \varepsilon_0/2).$$

Take k_0 such that $(p_{k_0} - 1)(\varepsilon_0/2) > \|\Phi\| + |\alpha| + (\varepsilon_0/2)$, and let $k > k_0$. If $[\omega] = [i_1 \cdots i_{(n+M)p_k}] \in F(\alpha; (n+M)p_k, \varepsilon_0/2)$, then there exists $x \in [\omega]$ such that

$$\left| \sum_{j=0}^{(n+M)p_k-1} \Phi(g^j x) - (n+M)p_k\alpha \right| \leqslant \frac{(n+M)p_k\varepsilon_0}{2}.$$

This implies that

$$\left| \sum_{j=0}^{n_k-1} \Phi(g^j x) - n_k\alpha \right| \leqslant \frac{(n+M)p_k\varepsilon_0}{2} + (n+M)(\|\Phi\| + |\alpha|) \leqslant n_k\varepsilon_0,$$

hence $[i_1 \cdots i_{n_k}] \in F(\alpha; n_k, \varepsilon_0)$ and

$$f(\alpha, s; (n+M)p_k, \varepsilon_0/2) \leqslant m^{(n+M)} f(\alpha, s; n_k, \varepsilon_0).$$

It follows that

$$m^{-(n+M)} C^{-p_k s} f(\alpha, s; n, \varepsilon_0/4)^{p_k} \leqslant f(\alpha, s; n_k, \varepsilon_0) \leqslant e^{n_k(\beta+\delta)}.$$

Therefore

$$f(\alpha, s; n, \varepsilon_0/4) \leqslant e^{(n_k)/(p_k)(\beta+\delta)} C^s m^{(n+M)/(p_k)}.$$

Letting $k \to \infty$, we have

$$f(\alpha, s; n, \varepsilon_0/4) \leqslant e^{(n+M)(\beta+2\delta)} C^s.$$

Letting $n \to \infty$, we have

$$\limsup_{n\to\infty} \frac{\log f(\alpha, s; n, \varepsilon_0/4)}{n} \leqslant \beta + 2\delta$$

as desired. Thus we have proved the equality in (4.2).

We will show below the upper semi-continuity of $P(\cdot, s)$. Given $\alpha \in L_\Phi$, for any $\eta > 0$, there is $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{\log f(\alpha, s; n, \varepsilon)}{n} < P(\alpha, s) + \eta.$$

Let $\beta \in L_\Phi$ with $|\beta - \alpha| < \frac{\varepsilon}{3}$. For each $[\omega] \in F(\beta; n, \varepsilon/3)$, there exists $x \in R_\omega$ such that $|\sum_{j=0}^{n-1} \Phi(g^j x) - n\beta| \leqslant \frac{n\varepsilon}{3}$. Hence

$$\left| \sum_{j=0}^{n-1} \Phi(g^j x) - n\alpha \right| \leqslant n\left(|\beta - \alpha| + \frac{\varepsilon}{3}\right) < n\varepsilon$$

and $[\omega] \in F(\alpha; n, \varepsilon)$. This proves that $F(\beta; n, \varepsilon/3) \subset F(\alpha; n, \varepsilon)$. It follows that $f(\beta, s; n, \varepsilon/3) \leqslant f(\alpha, s; n, \varepsilon)$; therefore

$$P(\beta, s) \leqslant \liminf_{n\to\infty} \frac{\log f(\beta, s; n, \varepsilon/3)}{n} \leqslant \liminf_{n\to\infty} \frac{\log f(\alpha, s; n, \varepsilon)}{n} \leqslant P(\alpha, s) + \eta.$$

This establishes the upper semi-continuity of $P(\cdot, s)$ at α.

The assertion on the Lipschitz property follows from the following inequality, which can be deduced from (4.1):

$$K_1 (\max_{x\in J} \|D_x g\|)^{-n} \leqslant \operatorname{diam}(R_\omega) \leqslant K_2 (\min_{x\in J} \|D_x g\|)^{-n}. \quad \blacksquare$$

For $\alpha \in L_\Phi$, we define $\Lambda(\alpha)$ to be the unique number s such that $P(\alpha, s) = 0$. Since $P(\alpha, s)$ is upper semi-continuous on α and strictly decreasing on s (by (4.3)), we have immediately

COROLLARY 4.4. *$\Lambda(\alpha) \geqslant 0$ and is an upper semi-continuous function on L_Φ.*

PROPOSITION 4.5. *For $\alpha \in L_\Phi$, we have*

$$\dim_H E(\alpha) = \dim_P E(\alpha) = \Lambda(\alpha).$$

Proof. We first show that for $\alpha \in L_\Phi$, $\dim_P E(\alpha) \leqslant \Lambda(\alpha)$. Let $\Lambda(\alpha) < t$; then $P(\alpha, t) < 0$; it follows that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{\log f(\alpha, t; n, \varepsilon)}{n} < -\delta.$$

Therefore there is a number $N = N(\varepsilon, \delta)$ such that for $n > N$, $f(\alpha, t; n, \varepsilon) \leqslant e^{-\delta n}$. Let

$$G(\alpha, k, \varepsilon) = \bigcap_{n=k}^{\infty} \left\{ x \in J : \left| \frac{1}{n} \sum_{i=0}^{n-1} \Phi(g^i x) - \alpha \right| < \varepsilon \right\}.$$

It is clear that for any $\varepsilon > 0$,

$$E(\alpha) \subset \bigcup_{k=1}^{\infty} G(\alpha, k, \varepsilon).$$

We show below that $\dim_P G(\alpha, k, \varepsilon) \leqslant t$ for each k. Let $\mathscr{F}$ be a collection of disjoint cylinders $[\omega]$, $\omega \in \Sigma_{A,n}$, $n \geqslant \max\{k, N\}$, such that the basic set R_ω has nonempty intersection with $G(\alpha, k, \varepsilon)$. Thus each $[\omega]$ in $\mathscr{F}$ is contained in $F(\alpha; n, \varepsilon)$ for some $n \geqslant N$. It follows that

$$\begin{aligned} \sum_{[\omega]\in\mathscr{F}} (\operatorname{diam} R_\omega)^t &\leqslant \sum_{n\geqslant N} \sum_{[\omega]\in F(\alpha;n,\varepsilon)} (\operatorname{diam} R_\omega)^t \\ &= \sum_{n\geqslant N} f(\alpha, t; n, \varepsilon) \leqslant \frac{e^{-\delta}}{1-e^{-\delta}} < \infty. \end{aligned}$$

Since the family of $[\omega] \in \mathscr{F}$ are disjoint, this implies that $\dim_P G(\alpha, k, \varepsilon) \leqslant t$. By the σ-stability of packing dimension, we have $\dim_P E(\alpha) \leqslant t$ as desired.

We now prove $\dim_H E(\alpha) \geqslant \Lambda(\alpha)$ for $\alpha \in L_\Phi$. Let $\Lambda(\alpha) > t$. By Proposition 4.3, there are $\ell_j \nearrow \infty$ and $\varepsilon_j \downarrow 0$ such that $f(\alpha, t; \ell_j, \varepsilon_j) > 1$. Write simply $F_{\ell_j} = F(\alpha; \ell_j, \varepsilon_j)$ and define a new sequence in the following manner:

$$\{\underbrace{\ell_1, \ldots, \ell_1}_{N_1}, \underbrace{\ell_2, \ldots, \ell_2}_{N_2}, \ldots, \underbrace{\ell_j, \ldots, \ell_j}_{N_j}, \ldots\},$$

where $N_j, j \geqslant 1$ diverge to ∞ fast and will be determined in the sequel. We relabel the sequence as $\{\ell_i^*\}$. Define

$$\mathscr{G} = \{R_{\bar{\omega}_1\cdots\bar{\omega}_k} : \ k \in \mathbb{N}, \ [\omega_i] \in F_{\ell_i^*} \text{ for } 1 \leqslant i \leqslant k\}$$

and

$$\Theta^* = \bigcap_{k\geqslant 1} \bigcup_{[\omega_i]\in F_{\ell_i^*}, 1\leqslant i\leqslant k} R_{\bar{\omega}_1\cdots\bar{\omega}_k},$$

where $\bar{\omega}_i$ is an extension of ω_i joining the leading letter of ω_{i+1} (with convention $\omega_{k+1} = \omega_1$). In view of Proposition 4.1, the collection $\mathcal{G}$ has the Moran structure and Θ^* is a Moran set in J. More precisely, Θ^* is constructed as follows. At level 0, we have the initial set $\bigcup_{i=1}^m R_i$. In step $k \geqslant 1$, we have the basic sets $R_{\bar{\omega}_1\cdots\bar{\omega}_k}$, $\omega_j \in F_{\ell_j^*}$ for $1 \leqslant j \leqslant k$. By (4.1), the maximal diameter M_k of the basic set $R_{\bar{\omega}_1\cdots\bar{\omega}_k}$ is less than $K_2(\lambda_{\min})^{-\ell_1^*-\cdots-\ell_k^*}$; the minimal contraction ratio ρ_k of the adjacent level is greater than $\frac{K_1}{2K_2}(\lambda_{\max})^{-\ell_k^*-M}$, where $\lambda_{\max} = \max_{x\in J}\|D_x g\|$, $\lambda_{\min} = \min_{x\in J}\|D_x g\|$. It follows that

$$\frac{\log \rho_k}{\log M_k} \leqslant \frac{(\ell_k^* + M)\log \lambda_{\max} - \log K_1 + \log (2K_2)}{(\ell_1^* + \cdots + \ell_k^*)\log \lambda_{\min} - \log K_2}.$$

In order for it to tend to zero (in view of Proposition 3.1), we can take $N_1 = 1$ and $N_j = 2^{l_{j+1}+N_{j-1}}$ for $j > 1$. Hence we conclude from Proposition 3.1 that

$$\dim_{\mathrm{H}} \Theta^* = \liminf_{k\to\infty} s_k,$$

where s_k satisfies the equation

$$\sum_{[\omega_i]\in F_{\ell_i^*}, 1\leqslant i\leqslant k} \mathrm{diam}(R_{\bar{\omega}_1\bar{\omega}_2\cdots\bar{\omega}_k})^{s_k} = 1. \tag{4.4}$$

Recall that we have proved in Lemma 4.2 that

$$\left(\frac{K_1}{2K_2\lambda_{\max}^M}\right)^k \prod_{i=1}^k \mathrm{diam}(R_{\omega_i}) \leqslant \mathrm{diam}(R_{\overline{\omega_1}\cdots\overline{\omega_k}}).$$

Making use of $\lim_{i\to\infty} \ell_i^* = \infty$ and $\mathrm{diam}(R_{\omega_i}) \leqslant K_2 \lambda_{\min}^{-\ell_i^*}$, we see that for any $\delta > 0$, there exists k_0 such that for $k > k_0$,

$$\left(\prod_{i=1}^k \mathrm{diam}(R_{\omega_i})\right)^{1+\delta} \leqslant \mathrm{diam}(R_{\overline{\omega_1}\cdots\overline{\omega_k}}).$$

Hence for $k > k_0$,

$$\begin{aligned}\prod_{i=1}^k f(\alpha, s_k(1+\delta); \ell_i^*, \varepsilon_i^*) &= \prod_{i=1}^k \sum_{[\omega_i]\in F_{\ell_i^*}} (\mathrm{diam}(R_{\omega_i}))^{s_k(1+\delta)} \\ &\leqslant \sum_{[\omega_i]\in F_{\ell_i^*},\, 1\leqslant i\leqslant k} (\mathrm{diam}(R_{\overline{\omega_1}\cdots\overline{\omega_k}}))^{s_k} = 1.\end{aligned}$$

Therefore $f(\alpha, s_k(1+\delta); \ell_i^*, \varepsilon_i^*) \leqslant 1$ for some $1 \leqslant i \leqslant k$. This implies that $s_k(1+\delta) \geqslant t$ since $f(\alpha, t; \ell_i^*, \varepsilon_i^*) \geqslant 1$. Therefore $\liminf_{k\to\infty} s_k \geqslant t$.

Now we prove that $\Theta^* \subset E(\alpha)$. Fix $x \in \Theta^*$. For a given large integer n, let k be the unique integer satisfying

$$\sum_{i=1}^{k-1} (\ell_i^* + M) \leqslant n < \sum_{i=1}^{k} (\ell_i^* + M).$$

By the definition of Θ^*, there exist $[\omega_i] \in F_{\ell_i^*}$, $i = 1, \ldots, k$, such that $x \in R_{\overline{\omega_1} \cdots \overline{\omega_k}}$. For $1 \leqslant i \leqslant k$, pick $x_i \in R_{\omega_i}$ such that

$$\left| \sum_{j=0}^{\ell_i^*-1} \Phi(g^i x_i) - \ell_i^* \alpha \right| < \varepsilon_i^* \ell_i^*.$$

Then

$$\begin{aligned}
&\left| \sum_{j=0}^{n-1} \Phi(g^j x) - n\alpha \right| \\
&\quad \leqslant \left| \sum_{i=1}^{k-1} \sum_{j=0}^{\ell_i^*-1} (\Phi(g^{\ell_1^* + \cdots + \ell_{i-1}^* + (i-1)M + j}(x)) - \alpha) \right| + (kM + \ell_k^*)(2\|\Phi\|) \\
&\quad \leqslant \left| \sum_{i=1}^{k-1} \sum_{j=0}^{\ell_i^*-1} (\Phi(g^{\ell_1^* + \cdots + \ell_{i-1}^* + (i-1)M + j}(x)) - \Phi(g^j x_i)) \right| \\
&\qquad + \left| \sum_{i=1}^{k-1} \sum_{j=0}^{\ell_i^*-1} (\Phi(g^j x_i) - \alpha) \right| + (kM + \ell_k^*)(2\|\Phi\|) \\
&\quad \leqslant \sum_{i=1}^{k-1} \left(\sum_{j=1}^{\ell_i^*-1} \|\Phi\|_j \right) + \sum_{i=1}^{k-1} (\varepsilon_i^* \ell_i^*) + (kM + \ell_k^*)(2\|\Phi\|).
\end{aligned}$$

Therefore

$$\frac{1}{n} \left| \sum_{j=0}^{n-1} \Phi(g^j x) - n\alpha \right| \leqslant \frac{\sum_{i=1}^{k-1} (\sum_{j=1}^{\ell_i^*-1} \|\Phi\|_j) + \sum_{i=1}^{k-1} (\varepsilon_i^* \ell_i^*) + (kM + \ell_k^*)(2\|\Phi\|)}{\sum_{i=1}^{k-1} \ell_i^*}.$$

Since $\lim_{i\to\infty} \ell_i^* = \infty$ and $\lim_{k\to\infty} \frac{\ell_k^*}{\sum_{i=1}^{k-1} \ell_i^*} = 0$ (by the definition of N_j), we have

$$\lim_{n\to\infty} \frac{1}{n} \left| \sum_{j=0}^{n-1} \Phi(g^j x) - n\alpha \right| = 0.$$

This proves that $x \in E(\alpha)$. Thus we have proved that $\Theta^* \subset E(\alpha)$, from which $\dim_{\mathrm{E}}(\alpha) \geqslant t$ follows. ∎

5. A VARIATIONAL PRINCIPLE

In this section, we will prove the following variational principle. Let $P(\alpha, s)$ be defined as in the last section.

THEOREM 5.1. *For any* $\alpha \in L_\Phi$ *and* $s \in \mathbb{R}$, *we have*

$$P(\alpha, s) = \max_{\mu \in \mathscr{F}_\Phi(\alpha)} \left\{ h_g(\mu) - s \int \log \|D_x g\| \, d\mu(x) \right\}.$$

Proof. *Part* I. Let μ be a g-invariant measure in $\mathscr{F}_\Phi(\alpha)$, i.e., $\alpha = \int \Phi \, d\mu$. We show that

$$P(\alpha, s) \geqslant h_g(\mu) - s \int \log\|D_x g\| \, d\mu(x).$$

By Proposition 2.6, there exists $v \in \mathscr{M}_\sigma(\Sigma_A)$ such that $\mu = v \circ \pi^{-1}$ and $h_g(v) = h_g(\mu)$. Let us assume at first that v is ergodic. Fix $\varepsilon > 0$. For any $n \in N$, let F_n $(\subseteq F(\alpha; n, \varepsilon))$ be the collection of all the n-cylinders $[\omega]$ in Σ_A such that there exists $y \in R_\omega$ satisfying

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \Phi(g^i y) - \alpha \right| < \varepsilon \tag{5.1}$$

and

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \log\|D_{g^i(y)} g\| - \int \log\|D_x g\| \, d\mu(x) \right| < \varepsilon. \tag{5.2}$$

We claim that the sequence $\{F_n\}$ satisfies

$$\liminf_{n \to \infty} \frac{\log \# F_n}{n} \geqslant h_g(\mu).$$

Indeed let $E_j = \bigcup \{[\omega] : \ [\omega] \in F_j\}$. By (5.1) and (5.2) and Birkhoff's ergodic theorem,

$$v\left(\bigcup_{n \geqslant 0} \bigcap_{j \geqslant n} E_j \right) = 1.$$

Hence there exists an integer N such that $\nu(\bigcap_{j\geqslant N} E_j) > 0$, which implies that $\underline{\dim}_{\mathrm{B}}(\bigcap_{j\geqslant N} E_j) \geqslant \dim_{\mathrm{H}} \nu = \frac{h_\sigma(\nu)}{\log m}$, and by the definition of box dimension,

$$\liminf_{n\to\infty} \frac{\log \# F_n}{n} \geqslant (\log m)\, \underline{\dim}_{\mathrm{B}}\left(\bigcap_{j\geqslant N} E_j\right) \geqslant h_\sigma(\nu) = h_g(\mu)$$

and the claim follows.

By (4.1) and (5.2), it is easy to see that for n large enough and any $\omega \in F_n$,

$$\left|\log \operatorname{diam}(R_\omega) + \int \log \|D_x g\|\, d\mu(x)\right| < 2\varepsilon. \tag{5.3}$$

Recall that $P(\alpha, s) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \log f(\alpha, s; n, \varepsilon)/n$ with $f(\alpha, s; n, \varepsilon) = \sum_{[\omega]\in F(\alpha;n,\varepsilon)} \operatorname{diam}(R_\omega)^s$. By applying the claim and (5.3), we have the desired inequality for $P(\alpha, s)$.

To complete the proof, we consider now ν without assuming the ergodicity. By Corollary 2.5, there exists a sequence of ergodic measures $\{\nu_n\}$ which converges to ν in the weak* topology and satisfies $\lim_{n\to\infty} h_\sigma(\nu_n) = h_\sigma(\nu)$. Let $\mu_n = \nu_n \circ \pi^{-1}$ and $\alpha_n = \int \Phi\, d\mu_n$; then

$$P(\alpha_n, s) \geqslant h_g(\mu_n) - s \int \log \|D_x g\|\, d\mu_n(x).$$

Letting $n \to \infty$, then $\alpha_n \to \alpha$, and by the upper semi-continuity of $P(\cdot, s)$, we have the desired inequality again.

Part II. In what follows, we show that for any $\alpha \in L_\Phi$ and $s \in \mathbb{R}$, there exists $\tilde{\mu} \in \mathscr{F}_\Phi(\alpha)$ such that

$$P(\alpha, s) \leqslant h_g(\tilde{\mu}) - s \int \log \|D_x g\|\, d\tilde{\mu}(x). \tag{5.4}$$

For each integer $k > 0$, we define two functions on Σ_A by

$$\Psi_k(\lambda) = \max_{y\in\pi(I_k(\lambda))} \Phi(y), \qquad \xi_k(\lambda) = \max_{y\in\pi(I_k(\lambda))} \log \|D_y g\|,$$

where $I_k(\lambda)$ denotes the k-cylinder which contains λ. It is clear that $\Psi_k(\lambda)$, $\xi_k(\lambda)$ depend only on the first k coordinates of λ. Since g is $C^{1+\delta}$, $\log \|D_x g\|$ is Hölder continuous. Since $\pi : \Sigma_A \to J$ is also Hölder continuous there exists $C > 0$ and $0 < \rho < 1$ such that

$$|\xi_k(\lambda) - \log \|D_{\pi(\lambda)} g\|\,| < C\rho^k.$$

To prove (5.4), we first observe that given any $\varepsilon > 0$, if $k \geqslant 1$ is sufficiently large,

$$|\Psi_k(\lambda) - \Phi(\pi(\lambda))| < \frac{\varepsilon}{2} \qquad \forall \lambda \in \Sigma_A.$$

Similarly to the definitions of $F(\alpha;n,\varepsilon)$ and $f(\alpha,s;n,\varepsilon)$ for Φ, we define

$$F_k(\alpha;n,\varepsilon) = \left\{[\omega] : \omega \in \Sigma_{A,n}, \ \left|\frac{1}{n}\sum_{j=0}^{n-1} \Psi_k(\sigma^j(\lambda)) - \alpha\right| < \varepsilon \text{ for some } \lambda \in [\omega]\right\}$$

and

$$f_k(\alpha,s;n,\varepsilon) = \sum_{[\omega]\in F_k(\alpha;n,\varepsilon)} \operatorname{diam}(R_\omega)^s.$$

It is clear that for any $s \in \mathbb{R}$ and sufficiently large k,

$$F(\alpha;n,\varepsilon/2) \subseteq F_k(\alpha;n,\varepsilon) \quad \text{and} \quad f(\alpha,s;n,\varepsilon/2) \leqslant f_k(\alpha,s;n,\varepsilon).$$

For $\omega \in \Sigma_{A,n+k-1}$ and $\lambda \in [\omega]$, by the definition of Ψ_k, we have $\Psi_k(\sigma^{j-1}\lambda) = \Psi_k(i_j \cdots i_{j+k-1})$ if $\omega = i_1 \cdots i_j \cdots i_{j+k} \cdots i_{n+k-1}$. We define an integer valued function $\phi_\omega : \Sigma_{A,k} \to \mathbb{Z}^+$ by

$$\phi_\omega(\tau) = \#\{k\text{-segments } i_j \cdots i_{j+k-1} \text{ of } \omega \text{ that equals } \tau\}, \qquad \tau \in \Sigma_{A,k}.$$

It is clear that $\sum_\tau \phi_\omega(\tau) = n$. Let $\mathscr{P}_k = \{\phi_\omega : \ \omega \in \Sigma_{A,n+k-1}, [\omega|n] \in F_k(\alpha,s;n,\varepsilon)\}$. Since $\phi_\omega(\tau) \leqslant n$ for each $\omega \in \Sigma_{A,n+k-1}$ and $\tau \in \Sigma_{A,k}$, $\#\mathscr{P}_k \leqslant n^{\#\Sigma_{A,k}} \leqslant n^{m^k}$. For each $\phi \in \mathscr{P}_k$, we let $\mathscr{T}(\phi)$ denote the collection of all the $\omega \in \Sigma_{A,n+k-1}$ such that $\phi = \phi_\omega$. Then for $\phi_\omega = \phi$ and $\lambda \in [\omega]$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \Psi_k(\sigma^j\lambda) = \frac{1}{n}\sum_{\tau\in\Sigma_{A,k}} \phi(\tau)\Psi_k(\tau). \tag{5.5}$$

The same is true if we replace Ψ_k by ξ_k. Also

$$\begin{aligned} f_k(\alpha,s;n,\varepsilon) &\leqslant \sum_{\phi\in\mathscr{P}_k}\left(\max_{\omega,\phi_\omega=\phi} \operatorname{diam}(R_{\omega|n})^s\right)\#\mathscr{T}(\phi) \\ &\leqslant n^{m^k} \max_{\phi\in\mathscr{P}_k}\left(\max_{\omega,\phi_\omega=\phi} \operatorname{diam}(R_{\omega|n})^s\right)\#\mathscr{T}(\phi). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\log f_k(\alpha,s;n,\varepsilon)}{n} \leqslant \max_{\phi\in\mathscr{P}_k}&\left(\frac{\log \#\mathscr{T}(\phi)}{n} + \frac{s}{n}\max_{\omega,\phi_\omega=\phi} \log \operatorname{diam}(R_{\omega|n})\right) \\ &+ O\left(\frac{\log n}{n}\right). \end{aligned} \tag{5.6}$$

(The constant in 'O' depends upon k.) For a fixed $\phi \in \mathscr{P}_k$, $\{\phi(\tau)/n\colon \tau \in \Sigma_{A,k}\}$ defines a probability vector. For the corresponding $\omega \in \Sigma_{A,n+k-1}$, if we let $n \to \infty$, the sequence of vectors $\{\phi(\cdot)/n\}$ has a weak*-accumulation point. By Lemma 3 in [13], this weak*-accumulation point belongs to Δ_k, and moreover for any $\eta > 0$, there is $N = N(\eta)$ such that when $n > N$, there exists a positive $p \in \Delta_k$ such that

$$\left|\frac{\phi(\tau)}{n} - p(\tau)\right| < \eta, \qquad p(\tau) > \frac{\eta}{m^{k+1}}, \qquad \tau \in \Sigma_{A,k}.$$

Consider the $(k-1)$-Markov measure ν_p defined by p (Proposition 2.3), for any cylinder $[\omega] \in \mathscr{T}(\phi)$ with $\omega = (x_i)_{i=1}^{n+k-1}$, $\phi_\omega = \phi$, we have

$$\nu_p([\omega]) = \frac{p(\omega|k)}{t(\omega|k)} \prod_{\tau\in\Sigma_{A,k}} t(\tau)^{\phi(\tau)} \geqslant \frac{\eta}{m^{k+1}} \prod_{\tau\in\Sigma_{A,k}} t(\tau)^{\phi(\tau)} := a,$$

where

$$t(\tau) = \frac{p(i_1, \ldots, i_k)}{\sum_\varepsilon p(i_1, \ldots, i_{k-1}, \varepsilon)}$$

for $\tau = i_1 \cdots i_k$. Then $a\#\mathscr{T}(\phi) \leqslant \nu_p(\bigcup_{[\omega]\in\mathscr{T}(\phi)}[\omega]) \leqslant 1$ and hence

$$\begin{aligned}\frac{\log \#\mathscr{T}(\phi)}{n} &\leqslant -\sum_\tau \frac{\phi(\tau)}{n}\log t(\tau) + O\left(\frac{|\log \eta|}{n}\right)\\ &\leqslant -\sum_\tau p(\tau)\log t(\tau) + O\left(\frac{|\log \eta|}{n} + \eta\,|\log \eta|\right)\\ &= h_\sigma(\nu_p) + O\left(\frac{|\log \eta|}{n} + \eta\,|\log \eta|\right).\end{aligned}$$

(The last equality follows by Proposition 2.2.)

For the second expression in (5.6), we have by (4.1),

$$\log \operatorname{diam}(R_{\omega|n}) \leqslant \log(2K_2) - \sum_{i=0}^{n-1} \log \|D_{g^i(x)}g\|$$

for any $x \in R_\omega$. Hence for any $\lambda \in [\omega]$,

$$\begin{aligned}
\frac{1}{n}\log \operatorname{diam}(R_{\omega|n}) &\leqslant \frac{1}{n}\log(2K_2) - \frac{1}{n}\sum_{i=0}^{n-1} \xi_k(\sigma^i(\lambda)) + C\rho^k \\
&\leqslant \frac{1}{n}\log(2K_2) + C\rho^k - \frac{1}{n}\sum_{\tau\in\Sigma_{A,k}} \phi_\omega(\tau)\xi_k(\tau) \qquad \text{(by (5.5))} \\
&\leqslant \frac{1}{n}\log(2K_2) + C\rho^k - \int \xi_k(y)\, d\nu_p(y) + m^k\eta\|\xi_k\| \\
&\leqslant \frac{1}{n}\log(2K_2) + 2C\rho^k + m^k\eta\|\xi_k\| - \int \log\|D_{\pi(y)}g\|\, d\nu_p(y).
\end{aligned}$$

Let $\Phi^*(\nu_p) \coloneqq \int \Phi\circ\pi\, d\nu_p$; then

$$\begin{aligned}
|\Phi^*(\nu_p) - \alpha| &\leqslant \left|\int \Psi_k\, d\nu_p - \alpha\right| + \varepsilon \\
&= \left|\sum_\tau p(\tau)\Psi_k(\tau) - \alpha\right| + \varepsilon \\
&\leqslant \left|\sum_\tau \frac{\phi(\tau)}{n}\Psi_k(\tau) - \alpha\right| + m^k\eta\|\Psi_k\| + \varepsilon \\
&= \left|\frac{1}{n}\sum_{j=0}^{n-1} \Psi_k(\sigma^j(\lambda)) - \alpha\right| + m^k\eta\|\Psi_k\| + \varepsilon \qquad \text{(by (5.5))} \\
&\leqslant m^k\eta\|\Psi_k\| + 2\varepsilon.
\end{aligned}$$

We conclude from (5.6) that

$$\begin{aligned}
\frac{\log f(\alpha, s, n, \varepsilon/2)}{n} &\leqslant \frac{\log f_k(\alpha, s, n, \varepsilon)}{n} \\
&\leqslant \sup\left(h_\sigma(\nu) - s\int \log\|D_{\pi(y)}g\|\, d\nu(y)\right) \\
&\quad + O\left(\frac{|\log\eta| + \log n}{n} + \eta\,|\log\eta|\right) + 2C\rho^k,
\end{aligned}$$

where the supremum is taken over $\nu \in \Delta_k$ and satisfies $|\Phi^*(\nu) - \alpha| < m^k\eta\|\Phi\|_k + 2\varepsilon$. Let $n \to \infty$ and then $\eta \to 0$; we get

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log f_k(\alpha, s; n, \varepsilon)}{n} \leqslant \sup_{|\Phi^*(\nu)-\alpha|\leqslant 3\varepsilon} &\left(h_\sigma(\nu) - s\int \log\|D_{\pi(y)}g\|\, d\nu(y)\right) \\
&+ 2C\rho^k.
\end{aligned}$$

Note that the set of invariant measures v on Σ_A such that $|\Phi_*(v) - \alpha| \leqslant 3\varepsilon$ is compact; by using the upper semi-continuity of $h_\sigma(v)$ and letting $\varepsilon \to 0$, then $k \to \infty$, we can find $\tilde{v}$ such that $\int \Phi \circ \pi \, d\tilde{v} = \alpha$ and

$$P(\alpha, s) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log f(\alpha, s; n, \varepsilon)}{n} \leqslant h_\sigma(\tilde{v}) - s \int \log \|D_{\pi(y)} g\| \, d\tilde{v}(y).$$

Take $\tilde{\mu} = \tilde{v} \circ \pi^{-1}$; then $\tilde{\mu} \in \mathscr{F}_\Phi(\alpha)$ and by Proposition 2.6,

$$P(\alpha, s) \leqslant h_g(\tilde{\mu}) - s \int \log \|D_x g\| \, d\tilde{\mu}(x).$$

This completes the proof. ∎

6. PROOF OF THE MAIN THEOREMS

From the results in the previous sections, we can conclude our first main theorem easily.

THEOREM 6.1. *Let J be a repeller of an expanding, $C^{1+\delta}$-conformal topological mixing map g.* Let *$\Phi : J \to \mathbb{R}^d$ be a continuous function. Then for any $\alpha \in L_\Phi$,*

$$\dim_{\mathrm{H}} E(\alpha) = \dim_{\mathrm{P}} E(\alpha) = \max_{\mu \in \mathscr{F}_\Phi(\alpha)} \frac{h_g(\mu)}{\int \log \|D_x g\| \, d\mu(x)}$$

Moreover $\dim_{\mathrm{H}} E(\alpha)$ *is an upper semi-continuous function of* α.

Proof. It follows from Proposition 4.5 that

$$\dim_{\mathrm{H}} E(\alpha) = \dim_{\mathrm{P}} E(\alpha) = \Lambda(\alpha),$$

where $\Lambda(\alpha) = s$ is the unique solution that satisfies $P(\alpha, s) = 0$. Thus by Theorem 5.1,

$$0 = \max_{\mu \in \mathscr{F}_\Phi(\alpha)} \{h_g(\mu) - \Lambda(\alpha) \int \log \|D_x g\| \, d\mu(x)\}.$$

It implies that

$$\Lambda(\alpha) = \max_{\mu \in \mathscr{F}_\Phi(\alpha)} \frac{h_g(\mu)}{\int \log \|D_x g\| \, d\mu(x)}$$

as is desired. The upper semi-continuity of $\dim_{\mathrm{H}} E(\alpha)$ follows from Corollary 4.4. ∎

We call a point $x \in J$ a *divergent point* if the limit $\alpha(x) = \lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} \Phi(g^j x)$ does not exist. In the following, we shall prove that the set of divergent points is of full dimension if it is not an empty set. Without loss of generality, assume that Φ takes real value (instead of the vector value); we let $\underline{\alpha}(x)$ and $\bar{\alpha}(x)$ be the lim inf and lim sup of $n^{-1}\sum_{j=0}^{n-1}\Phi(g^jx)$. For any $\beta<\alpha$, set

$$\tilde{E}(\alpha)=\left\{x\in J:\quad \lim_{k\to\infty}\frac{1}{n_k}\sum_{j=0}^{n_k-1}\Phi(g^jx)=\alpha \text{ for some } \{n_k\}_{k=1}^{\infty}\uparrow\infty\right\},$$

$$\bar{E}(\alpha)=\{x\in J:\ \bar{\alpha}(x)=\alpha\},$$

$$\underline{E}(\alpha)=\{x\in J:\ \underline{\alpha}(x)=\alpha\},$$

$$\tilde{E}(\alpha,\beta)=\{x\in J:\ \bar{\alpha}(x)=\alpha,\ \underline{\alpha}(x)=\beta\}.$$

Lemma 6.2. *For any* $\alpha\in L_\Phi$, $\dim_{\mathrm{H}}\tilde{E}(\alpha)\leqslant\Lambda(\alpha)$.

Proof. Assume that $\Lambda(\alpha)<t$; we will show that $\dim_{\mathrm{H}}\tilde{E}(\alpha)\leqslant t$. By the strict decreasing property of $P(\alpha,\cdot)$, we have $P(\alpha,t)<0$. Hence there exists $\varepsilon>0$ and $\delta>0$ such that

$$\limsup_{n\to\infty}\frac{\log f(\alpha,t;n,\varepsilon)}{n}<-\delta.$$

This implies $f(\alpha,t;n,\varepsilon)<\exp(-n\delta)$ for $n\geqslant N_0$. Since

$$\begin{aligned}\tilde{E}(\alpha)&\subseteq\bigcap_{k=0}^{\infty}\bigcup_{n\geqslant k}\left\{x\in J:\ \left|\frac{1}{n}\sum_{j=0}^{n-1}\Phi(g^jx)-\alpha\right|<\varepsilon\right\}\\&\subseteq\bigcap_{k=N_0}^{\infty}\bigcup_{n\geqslant k}\left\{x\in J:\ \left|\frac{1}{n}\sum_{j=0}^{n-1}\Phi(g^jx)-\alpha\right|<\varepsilon\right\},\end{aligned}$$

for each $k>N_0$, the collection

$$\mathscr{G}_\ell=\{R_\omega:\ [\omega]\in F(\alpha;n,\varepsilon)\text{ for some } n\geqslant\ell\}$$

is a cover of $\tilde{E}(\alpha)$. Note that

$$\begin{aligned}\sum_{R_\omega\in\mathscr{G}_\ell}(\operatorname{diam}R_\omega)^t&=\sum_{n\geqslant\ell}\ \sum_{[\omega]\in F(\alpha;n,\varepsilon)}(\operatorname{diam}R_\omega)^t\\&=\sum_{n\geqslant\ell}f(\alpha,t;n,\varepsilon)\leqslant\frac{\exp(-\delta)}{1-\exp(-\delta)}<\infty\end{aligned}$$

for each ℓ, and we have

$$\mathscr{H}^t(\tilde{E}(\alpha)) \leqslant \frac{\exp(-\delta)}{1-\exp(-\delta)} < \infty.$$

It follows that $\dim_{\mathrm{H}} \tilde{E}(\alpha) \leqslant t$. ∎

From Theorem 6.1 and Lemma 6.2, we have the following immediately.

PROPOSITION 6.3. *For any* $\alpha \in L_\Phi$,

$$\dim_{\mathrm{H}} \bar{E}(\alpha) = \dim_{\mathrm{H}} \underline{E}(\alpha) = \dim_{\mathrm{H}} \tilde{E}(\alpha) = \Lambda(\alpha).$$

Remark. We point out that by modifying Lemma 6.2, we can prove $\dim_{\mathrm{P}} \bigcap_{\alpha \in L_\Phi} \tilde{E}(\alpha) = \dim_{\mathrm{H}} J$. For simplicity we omit the proof since we do not use this result here.

PROPOSITION 6.4. *For any* $\alpha \in L_\Phi$ *and* $\beta \in L_\Phi$ *satisfying* $\beta < \alpha$,

$$\dim_{\mathrm{H}} \tilde{E}(\alpha, \beta) = \min(\Lambda(\alpha), \Lambda(\beta)).$$

Proof. By Lemma 6.2, $\dim_{\mathrm{H}} \tilde{E}(\alpha, \beta) \leqslant \min(\Lambda(\alpha), \Lambda(\beta))$ is obvious. For the reverse inequality, we use the similar idea in the proof of the lower bound in Proposition 4.5; we will only give a sketch here.

For any $t < \min(\Lambda(\alpha), \Lambda(\beta))$, by Proposition 4.3, there are $\ell_j \uparrow \infty$ and $\varepsilon_j \downarrow 0$ such that $f(\alpha, t; \ell_{2j-1}, \varepsilon_{2j-1}) > 1$, $f(\beta, t; \ell_{2j}, \varepsilon_{2j}) > 1$, $j = 1, 2, \ldots$. Write simply $F_{\ell_{2j-1}} = F(\alpha; \ell_{2j-1}, \varepsilon_{2j-1})$, $F_{\ell_{2j}} = F(\beta; \ell_{2j}, \varepsilon_{2j})$, $j = 1, 2, \ldots$, and define a new sequence $\{\ell^*{}_j\}$ in the following manner:

$$\underbrace{\ell_1, \ldots, \ell_1}_{N_1}; \underbrace{\ell_2, \ldots, \ell_2}_{N_2}; \ldots; \underbrace{\ell_j, \ldots, \ell_j}_{N_j}; \ldots,$$

where $N_j, j \geqslant 1$ diverge to ∞ fast. By the same proof as in Proposition 4.5, we can show that the corresponding Moran set Θ^* is contained in $\tilde{E}(\alpha, \beta)$ and $\dim_{\mathrm{H}} \tilde{E}(\alpha, \beta) \geqslant \dim_{\mathrm{H}} \Theta^* = \liminf s_k \geqslant t$. ∎

LEMMA 6.5. *If* $\lim_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \Phi(g^j x) = \alpha$ *for some* $x \in J$ *and* $\{n_k\}_{k=1}^\infty \nearrow \infty$, *then* $\alpha \in L_\Phi$.

Proof. For each $k \in \mathbb{N}$, let $\varepsilon_k = |\frac{1}{n_k} \sum_{j=0}^{n_k - 1} \Phi(g^j x) - \alpha|$ and pick $\omega_k \in \Sigma_{A, n_k}$ such that $x \in R_{\omega_k}$. We define a new sequence in the following manner:

$$\{\underbrace{\omega_1, \ldots, \omega_1}_{N_1}, \underbrace{\omega_2, \ldots, \omega_2}_{N_2}, \ldots, \underbrace{\omega_j, \ldots, \omega_j}_{N_j}, \ldots\},$$

where $N_1 = 1$ and $N_j = 2^{n_{j+1} + N_{j-1}}$, $j \geqslant 2$. We relabel this sequence as $\{\omega_i^*\}$, and define $\{\ell_i^*\}$ as the length sequence of $\{\omega_i^*\}$. Take $y \in \bigcap_{k=1}^\infty R_{\overline{\omega_1^* \cdots \omega_k^*}}$, we show below that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\Phi(g^j y)=\alpha,$$

which implies $\alpha\in L_\Phi$.

For a large integer n, there is one unique integer k satisfying

$$\sum_{i=1}^{k-1}(\ell_i^*+M)\leqslant n<\sum_{i=1}^{k}(\ell_i^*+M).$$

Then

$$\begin{aligned}
&\left|\sum_{j=0}^{n-1}\Phi(g^j y)-n\alpha\right|\\
&\quad\leqslant\left|\sum_{i=1}^{k-1}\sum_{j=0}^{\ell_i^*-1}(\Phi(g^{\ell_1^*+\cdots+\ell_{i-1}^*+(i-1)M+j}(y))-\alpha)\right|+(kM+\ell_k^*)(2\|\Phi\|)\\
&\quad\leqslant\left|\sum_{i=1}^{k-1}\sum_{j=0}^{\ell_i^*-1}(\Phi(g^{\ell_1^*+\cdots+\ell_{i-1}^*+(i-1)M+j}(y))-\Phi(g^j x))\right|\\
&\qquad\qquad+\left|\sum_{i=1}^{k-1}\sum_{j=0}^{\ell_i^*-1}(\Phi(g^j x)-\alpha)\right|+(kM+\ell_k^*)(2\|\Phi\|)\\
&\quad\leqslant\sum_{i=1}^{k-1}\left(\sum_{j=1}^{\ell_i^*-1}\|\Phi\|_j\right)+\sum_{i=1}^{k-1}(\varepsilon_i^*\ell_i^*)+(kM+\ell_k^*)(2\|\Phi\|).
\end{aligned}$$

Since $\lim_{i\to\infty}\ell_i^*=\infty$ and $\lim_{k\to\infty}\frac{\ell_k}{\sum_{i=1}^{k-1}\ell_i^*}=0$ (by the definition of N_j), we have

$$\lim_{n\to\infty}\frac{1}{n}\left|\sum_{j=0}^{n-1}\Phi(g^j y)-n\alpha\right|=0,$$

which implies $\alpha\in L_\Phi$. ∎

THEOREM 6.6. *Let J be a repeller of an expanding, $C^{1+\delta}$-conformal topological mixing map g. Let $\Phi: J\to\mathbb{R}^d$ be a continuous function. Then either*

(i) *all points $x\in J$ have the same ergodic limit; or*

(ii) *the set of points x such that the limit defining $\alpha(x)$ does not exist is of the same Hausdorff dimension as that of J.*

Proof. We can assume, without loss of generality, that Φ takes real value. Let $P_J(\phi)$ be the standard topological pressure with respect to g where the potential ϕ is a continuous function on J [21, Chap. 4]. If we consider $\phi(x) = s \log \|D_x g\|, s \in \mathbb{R}$, then Bowen's equation states that there exists a unique t such that

$$P_J(-t \log \|D_x g\|) = 0$$

(see [21, Appendix II]). Since $g: M \to M$ is a $C^{1+\delta}$-conformal expanding map, there is a unique equilibrium measure μ_1 corresponding to the Hölder continuous potential function $-t \log \|D_x g\|$ and

$$\dim_H J = \dim_H \mu_1 = \frac{h_g(\mu_1)}{\int \log \|D_x g\| \, d\mu_1(x)} \tag{6.1}$$

[21, Theorem 20.1]. Set $\alpha = \int \Phi(x) \, d\mu_1(x)$. If (i) of the theorem does not hold, we can assume, without loss of generality, that there exists $x \in J$, $\beta < \alpha$ such that $\underline{\alpha}(x) = \beta$. By Lemma 6.5, $\beta \in L_\Phi$. Thus by Proposition 6.3, there exists $\mu_2 \in \mathcal{M}_g(J)$ with $\int \Phi(x) \, d\mu_2(x) = \beta$ and

$$\dim_H \tilde{E}(\beta) = \frac{h_g(\mu_2)}{\int \log \|D_x g\| \, d\mu_2(x)}. \tag{6.2}$$

For any $\delta > 0$, consider $\tilde{E}(\alpha, (1-\delta)\alpha + \delta\beta)$. It is clear that

$$\tilde{E}(\alpha, (1-\delta)\alpha + \delta\beta) \subseteq \{x \in J : \ \underline{\alpha}(x) < \bar{\alpha}(x)\}.$$

Let us denote the second set by F. By Proposition 6.4, $\dim_H \tilde{E}(\alpha, (1-\delta)\alpha + \delta\beta) = \min\{\Lambda(\alpha), \ \Lambda((1-\delta)\alpha + \delta\beta)\}$ and by Proposition 4.5 and Theorem 6.1,

$$\begin{aligned}\Lambda((1-\delta)\alpha + \delta\beta) &\geqslant \frac{h_g((1-\delta)\mu_1 + \delta\mu_2)}{\int \log \|D_x g\| \, d((1-\delta)\mu_1(x) + \delta\mu_2(x))} \\ &= \frac{(1-\delta)h_g(\mu_1) + \delta h_g(\mu_2)}{\int \log \|D_x g\| \, d((1-\delta)\mu_1(x) + \delta\mu_2(x))}.\end{aligned}$$

Let $\delta \to 0$; we have $\dim_H F \geqslant \dim_H(\mu_1) = \dim_H(J)$. This completes the proof of Theorem 6.6. ∎

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