

**LYAPUNOV EXPONENTS FOR PRODUCTS OF MATRICES AND  
MULTIFRACTAL ANALYSIS.  
PART II: GENERAL MATRICES**

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ABSTRACT. We continue the study in [11, 14] on the upper Lyapunov exponents for products of matrices. Here we consider general matrices. In general, the variational formula about Lyapunov exponents we obtained in part I does not hold in this setting. Anyway we focus our interest on a special case where the matrix function  $M(x)$  takes finite values  $M_1, \dots, M_m$ . In this case we prove the variational formula under an additional irreducibility condition. This extends a previous result of the author and Lau [14]. As an application, we prove a new multifractal formalism for a certain class of self-similar measures on  $\mathbb{R}$  with overlaps. More precisely, let  $\mu$  be the self-similar measure on  $\mathbb{R}$  generated by a family of contractive similitudes  $\{S_j = \rho x + b_j\}_{j=1}^\ell$  which satisfies the finite type condition. Then we can construct a family (finite or countably infinite) of closed intervals  $\{I_j\}_{j \in \Lambda}$  with disjoint interiors, such that  $\mu$  is supported on  $\bigcup_{j \in \Lambda} I_j$  and the restricted measure  $\mu|_{I_j}$  of  $\mu$  on each interval  $I_j$  satisfies the complete multifractal formalism. Moreover the dimension spectrum  $\dim_H E_{\mu|_{I_j}}(\alpha)$  is independent of  $j$ .

1. INTRODUCTION

The present paper is a continuation of our work in [11, 14] for studying the upper Lyapunov exponents for products of matrices. Here we consider general matrices. An application will be given to multifractal analysis of self-similar measures with overlaps.

First we recall some basic notation. Let  $\sigma$  be the shift map on  $\Sigma = \{1, 2, \dots, m\}^{\mathbb{N}}$  ( $m \geq 2$  an integer). Endow  $\Sigma$  with the metric  $d(x, y) = m^{-n}$  for  $x = (x_j)_{j=1}^\infty$  and  $y = (y_j)_{j=1}^\infty$  where  $n$  is the largest integer such that  $x_j = y_j$  ( $1 \leq j \leq n$ ). Let  $M$  be a continuous function defined on  $\Sigma$  taking values in the set of  $d \times d$  matrices with non-negative entries. The *upper Lyapunov exponent*  $\lambda_M(x)$  of  $M$  at  $x$  is defined by

$$(1.1) \quad \lambda_M(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|M(x)M(\sigma x) \dots M(\sigma^{n-1}x)\|,$$

when the limit exists. Here  $\|\cdot\|$  denotes the matrix norm defined by  $\|A\| := \mathbf{1}^T A \mathbf{1}$ , where  $\mathbf{1}$  is the  $d$ -dimensional column vector each coordinate of which is 1. The *pressure function* of  $M$  is defined by

$$P_M(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in \Sigma_n} \sup_{x \in [\omega]} \|M(x)M(\sigma x) \dots M(\sigma^{n-1}x)\|^q, \quad q \in \mathbb{R},$$

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where  $\Sigma_n$  denotes the set of all words of length  $n$  over  $\{1, \dots, m\}$ ; for  $\omega = \omega_1 \dots \omega_n \in \Sigma_n$ ,  $[\omega]$  denotes the cylinder set  $\{x = (x_j)_{j=1}^{\infty} \in \Sigma : x_j = \omega_j, 1 \leq j \leq n\}$ . The limit in the above definition always exists for  $q > 0$ . This can be shown by a subadditive argument. With some additional conditions on  $M$  (e.g.,  $M$  is strictly positive), the limit exists for  $q \in \mathbb{R}$ . Let  $\mathcal{M}_\sigma(\Sigma)$  be the set of all  $\sigma$ -invariant Borel probability measures on  $\Sigma$ . The matrix function  $M$  induces a map  $M_*: \mathcal{M}_\sigma(\Sigma) \rightarrow \mathbb{R} \cup \{-\infty\}$  given by

$$M_*(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|M(y)M(\sigma y) \dots M(\sigma^{n-1}y)\| d\mu(y), \quad \mu \in \mathcal{M}_\sigma(\Sigma).$$

The limit exists because of the subadditivity of the integral. In [18] Furstenberg and Kesten proved that for each ergodic measure  $\mu$  on  $\Sigma$ ,

$$\lambda_M(x) = M_*(\mu), \quad \mu \text{ a.s. } x \in \Sigma.$$

To study the point-wise property of  $\lambda_M(\cdot)$  more delicately, we define

$$(1.2) \quad L_M = \{\alpha \in \mathbb{R} : \alpha = \lambda_M(x) \text{ for some } x \in \Sigma\}$$

and

$$(1.3) \quad E_M(\alpha) = \{x \in \Sigma : \lambda_M(x) = \alpha\} \quad (\alpha \in L_M).$$

The following theorem was proved in [11]:

**Theorem A** ([11]). *Suppose  $M$  is a continuous function defined on  $\Sigma$  taking values in the set of  $d \times d$  matrices with strictly positive entries. Then  $L_M$  is an interval and for any  $\alpha \in L_M$ ,*

$$\begin{aligned} \dim_H E_M(\alpha) &= \dim_P E_M(\alpha) \\ &= \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha q + P_M(q)\} \\ &= \frac{1}{\log m} \sup \{h(\mu) : \mu \in \mathcal{M}_\sigma(\Sigma), M_*(\mu) = \alpha\}, \end{aligned}$$

where  $\dim_H$  and  $\dim_P$  denote the Hausdorff dimension and the packing dimension, respectively, and  $h(\mu)$  denotes the measure-theoretic entropy of  $\mu$ . Moreover,  $\dim_H E_M(\alpha)$  is a concave and continuous function of  $\alpha$  on  $L_M$ .

Theorem A extends the corresponding results for the Birkhoff averages of scalar functions, for which one is referred to, e.g., [1, 2, 3, 9, 10, 16, 35, 38, 40, 41, 47]. However since Theorem A depends very much on the positivity of  $M$ , for certain purposes of application (e.g., the multifractal analysis of self-similar measures with overlaps) we would like to consider the case that the matrix function  $M$  is only assumed to be non-negative. However under this general non-negative assumption, the result of Theorem A is no longer valid. The following is a counter-example.

**Example 1.1** Set  $\Sigma = \{1, 2, 3, 4\}^{\mathbb{N}}$ . Let the matrix function  $M(x)$  on  $\Sigma$  be defined as  $M(x) = M_{x_1}$  for  $x = (x_j)_{j=1}^{\infty}$ , where  $M_i$  ( $1 \leq i \leq 4$ ) are four diagonal matrices given by

$$M_1 = M_2 = \text{diag}(1, 2, 0, 0), \quad M_3 = \text{diag}(1, 0, 3, 0), \quad M_4 = \text{diag}(1, 0, 0, 4).$$

It is easily checked that

$$P_M(q) = \begin{cases} q \log 4, & \text{if } q \geq 1, \\ \log 4, & \text{otherwise.} \end{cases}$$

However  $L_M = \{0, \log 2, \log 3, \log 4\}$ , and

$$\dim_H E_M(\log 3) = 0 < 1 - \frac{\log 3}{\log 4} = \frac{1}{\log 4} \inf_{q \in \mathbb{R}} \{-q \log 3 + P_M(q)\}.$$

For the above reason, we shall impose more conditions on  $M$ :

(H1)  $M(x) = M_i$  if  $x \in [i]$ ,  $i = 1, \dots, m$ ;

(H2)  $M$  is irreducible in the following sense: there exists  $r > 0$  such that

$$(1.4) \quad \sum_{k=1}^r \left( \sum_{i=1}^m M_i \right)^k > \mathbf{0}.$$

In this case, the pressure function  $P_M(q)$  can be defined by

$$(1.5) \quad P_M(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{J \in \mathcal{N}_n} \|M_J\|^q, \quad \forall q \in \mathbb{R},$$

where  $\mathcal{N}_n = \{J \in \Sigma_n : M_J \neq \mathbf{0}\}$  and  $M_J = M_{j_1} \cdots M_{j_n}$ . It was proved in [14] that the limit exists for any  $q \in \mathbb{R}$  (we include a proof in section 2 for the convenience of the reader). The first result of this paper is the following:

**Theorem 1.1.** *Suppose that  $M$  is a function on  $\Sigma$  taking values in the set of all  $d \times d$  non-negative matrices. Assume  $M$  satisfies the conditions (H1) and (H2). Then  $L_M$  is an interval and for any  $\alpha \in L_M$ ,*

$$(1.6) \quad \dim_H E_M(\alpha) = \dim_P E_M(\alpha) = \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha q + P_M(q)\}.$$

Moreover  $\dim_H E_M(\alpha)$  is a concave and continuous function of  $\alpha$  on  $L_M$ .

Under the setting of this theorem, some partial results were obtained by Feng and Lau in [14]. It was shown that for each  $q > 0$ , there exists a unique ergodic probability measure  $\mu_q$  on  $\Sigma$  satisfying the following Gibbs property:

$$(1.7) \quad C^{-1} \leq \frac{\mu_q([J])}{\exp(-nP_M(q)) \cdot \|M_J\|^q} \leq C, \quad \forall n \in \mathbb{N}, J \in \Sigma_n$$

where  $C > 0$  is a constant (see [14, Theorem 3.2]). The measure  $\mu_q$  on  $\Sigma$  is called the *Gibbs measure* associated with  $M$  and  $q$ . Using this Gibbs property, we proved that  $P_M(q)$  is always differentiable on  $(0, \infty)$  and (1.6) holds for any  $\alpha = P'_M(q)$  with  $q > 0$  (see [14, Theorems 3.3 and 3.4]). Unfortunately, the existence of the Gibbs measure can not be extended to the case  $q < 0$ ; actually we can construct a matrix function  $M$  satisfying (H1) and (H2) such that  $P_M(q)$  is not differentiable at some point  $q < 0$  (see Example 6.6). Hence some new idea is needed to prove (1.6) for all  $\alpha \in L_M$ .

Now we outline the idea for proving Theorem 1.1. Under the condition of the theorem, there is a constant  $C > 0$  such that for any  $I \in \Sigma_n$  and  $J \in \Sigma_\ell$  there exists  $K \in \bigcup_{k=1}^r \Sigma_k$  such that  $\|M_{IKJ}\| \geq C \|M_I\| \|M_J\|$ . Using this property and modifying our proof in [11] delicately, we set up a formal formula for  $\dim_H E_M(\alpha)$  similar to that in [11]. To show the variational relationship between  $\dim_H E_M(\alpha)$  and  $P_M(q)$ , we use a new idea. In fact we first give such a relationship between  $\dim_H E_M^{(\ell)}(\alpha)$  and  $P_M(\ell, q)$  ( $\ell = 1, 2, \dots$ ), where

$$E_M^{(\ell)}(\alpha) = \left\{ (x_i)_{i=1}^\infty \in \Sigma : \lim_{n \rightarrow \infty} \frac{1}{n\ell} \log \left( \prod_{j=1}^n \|M_{x_{(j-1)\ell+1} x_{(j-1)\ell+2} \cdots x_{j\ell}}\| \right) = \alpha \right\}, \quad \alpha \in \mathbb{R}$$

and  $P_M(\ell, q) = \frac{1}{\ell} \log (\sum_{J \in \mathcal{N}_\ell} \|M_J\|^q)$ . Then we prove the theorem by showing that  $\dim_H E_M^{(\ell)}(\alpha)$  and  $P_M(\ell, q)$  converge to  $\dim_H E_M(\alpha)$  and  $P_M(q)$  respectively, as  $\ell$  tends to infinity.

We point out that an essentially identical proof shows that the result of Theorem 1.1 remains true if we remove the non-negativity assumption of  $\{M_i\}_{i=1}^m$ , but instead of (H2) we assume directly the more general condition (H2)': *there exist a constant  $C > 0$  and  $r \in \mathbb{N}$  such that for any  $I \in \Sigma_n$  and  $J \in \Sigma_\ell$  there exists  $K \in \bigcup_{k=1}^r \Sigma_k$  such that  $\|M_{IKJ}\| \geq C\|M_I\|\|M_J\|$* . Here we use the standard matrix norm. For instance, this assumption (H2)' is fulfilled if the family  $\{M_i\}_{i=1}^m$  of  $d \times d$  complex matrices satisfies the following type of irreducibility: there is no proper non-zero linear subspace  $V$  of  $\mathbb{C}^d$  such that  $M_i(V) \subseteq V$  for any  $1 \leq i \leq m$  (cf. [6, p. 48]). For a proof, see Proposition 2.8. In this situation, the left endpoint of  $L_M$  may be  $-\infty$ .

Theorem 1.1 has an important application in the multifractal analysis for a class of self-similar measures with overlaps. Actually, this is part of our original motivation for developing the multifractal theory for products of matrices. Before stating the result, we recall some notation and background. Let  $\nu$  be a finite Borel measure on  $\mathbb{R}^n$  with compact support. For  $q \in \mathbb{R}$ , the  $L^q$ -spectrum of  $\nu$  is defined by

$$\tau(q) = \tau(\nu, q) = \liminf_{\delta \downarrow 0} \frac{\log(\sup \sum_i \nu(B_\delta(x_i))^q)}{\log \delta},$$

where the supremum is taken over all the families of disjoint balls  $B_\delta(x_i)$  of radius  $\delta$  and center  $x_i \in \text{supp}(\nu)$ . Denote by

$$L_\nu = \left\{ \alpha \in \mathbb{R} : \lim_{\delta \downarrow 0} \frac{\log \nu(B_\delta(x))}{\log \delta} = \alpha \text{ for some } x \in \text{supp}(\nu) \right\}$$

and

$$E_\nu(\alpha) = \left\{ x \in \text{supp}(\nu) : \lim_{\delta \downarrow 0} \frac{\log \nu(B_\delta(x))}{\log \delta} = \alpha \right\}.$$

We say that  $\nu$  satisfies the *complete multifractal formalism* if the following two conditions are satisfied:

(i)  $L_\nu = [\alpha_{\min}, \alpha_{\max}]$ , where

$$\alpha_{\min} = \lim_{q \rightarrow +\infty} \frac{\tau(q)}{q}, \quad \alpha_{\max} = \lim_{q \rightarrow -\infty} \frac{\tau(q)}{q}.$$

(ii) For any  $\alpha \in L_\nu$ ,  $\dim_H E_\nu(\alpha) = \inf\{\alpha q - \tau(q) : q \in \mathbb{R}\}$ .

For the framework of multifractal analysis of general measures, one may see, e.g., [4, 7, 8, 21, 24, 25, 26, 37, 39, 42, 44]. It is well known that if  $\mu$  is the self-similar measure defined by a family of contractive similitudes  $\{S_j\}_{j=1}^\ell$  which satisfies the *open set condition* [28],  $\tau(q)$  can be calculated by an explicit analytic formula and  $\mu$  satisfies the complete multifractal formalism (see [8, 37]). However, if the family  $\{S_j\}_{j=1}^\ell$  does not satisfy any separation condition, it is much hard to obtain a formula for  $\tau(q)$  and there are few results on the validity of the multifractal formalism. Some partial results have been obtained in the case that the family  $\{S_j\}_{j=1}^\ell$  satisfies some separation conditions weaker than the open set condition. In [31] Lau and Ngai introduced a *weak separation condition*, which is strictly weaker than the open set condition. They proved that a partial multifractal formalism will hold under this condition, i.e.,  $\dim_H E_\mu(\alpha) = \inf\{\alpha q - \tau(q) : q \in \mathbb{R}\}$  for all those  $\alpha \in \{\tau'(t) : t > 0 \text{ and } \tau'(t) \text{ exists}\}$ . It is unknown whether  $\tau$  is always differentiable for  $t > 0$  under the weak separation condition. Nevertheless, in [12], the author proved that  $\tau(q)$  is differentiable on  $(0, \infty)$  whenever the family  $\{S_j\}_{j=1}^\ell$  has the same contractive ratio on  $\mathbb{R}$  and satisfies the *finite type condition*. Recall that a family of similitudes

$$S_j(x) = \rho x + b_j, \quad 0 < \rho < 1, \quad b_j \in \mathbb{R}, \quad j = 1, \dots, \ell$$

satisfies the finite type condition if there is a finite set  $\Gamma \subset \mathbb{R}^+$  such that for each integer  $n > 0$  and any two indices  $J = j_1 \dots j_n$  and  $J' = j'_1 \dots j'_n$ ,

$$(1.8) \quad \text{either} \quad \rho^{-n}|S_J(0) - S_{J'}(0)| > c \quad \text{or} \quad \rho^{-n}|S_J(0) - S_{J'}(0)| \in \Gamma,$$

where  $S_J$  denotes the composition  $S_{j_1} \circ \dots \circ S_{j_n}$  and

$$c = (1 - \rho)^{-1} \left( \max_{1 \leq j \leq \ell} b_j - \min_{1 \leq i \leq \ell} b_i \right).$$

The finite type condition was introduced by Ngai and Wang [33]. It is substantially weaker than the open set condition, but a little stronger than the weak separation condition (see [34]). The finite type condition is satisfied by many interesting overlap cases, such as the Bernoulli convolutions associated with Pisot numbers. Combining the results in [12] and [31], under the finite type condition, we have  $\dim_H E_\mu(\alpha) = \inf\{\alpha q - \tau(q) : q \in \mathbb{R}\}$  for all those  $\alpha \in \{\tau'(t) : t > 0\}$ . Recently this result was extended by Testud [49] to a special class of self-similar measures which satisfy the weak separation condition. Now a natural question arises as to whether or not the complete multifractal formalism holds in the setting of the finite type condition. To answer the question it suffices to consider the case  $t < 0$ . Some special examples have been considered towards this direction. In [13], the author proved that for the Erdős measure  $\mu$  (i.e., the Bernoulli convolution associated with the golden ratio), the function  $\tau(q)$  is not differentiable at some point  $q < 0$ . It is rather surprising. Nevertheless, the multifractal formalism (for all  $\alpha \in \text{int}(L_\mu)$ ) of such a measure is proved to be valid by Feng and Olivier [17], which extended the partial results in [30, 43]. For Bernoulli convolutions associated with other Pisot numbers, see [13] and [36]. In [27] Hu and Lau studied another interesting measure  $\mu$ , which is the three-fold convolution of the standard Cantor measure, and they found that the set  $L_\mu$  is not an interval (actually it is a union of an interval and an isolated point). That is also a very strange fact (it shows that the complete multifractal formalism can *fail* for a self-similar measure satisfying the finite type condition). Later on this measure was extensively studied in [15, 32]. Similarly its  $L^q$ -spectrum  $\tau(q)$  has a non-differentiable point in  $(-\infty, 0)$  [32], whilst an adjusted multifractal formalism still holds [15]. More interesting examples with similar phenomena were found and studied by Shmerkin [46] and Testud [49]. For a general self-similar measure  $\mu$  on  $\mathbb{R}$  satisfying the finite type condition, the author proved in [12] that for a class of intervals  $J$ ,  $\mu(J)$  are controlled by products of a family of non-negative matrices (see Proposition 5.1 and §6 for details). Applying this result and Theorem 1.1, we will prove the following general result:

**Theorem 1.2.** *Let  $\mu$  be the self-similar measure on  $\mathbb{R}$  satisfying*

$$(1.9) \quad \mu = \sum_{j=1}^{\ell} p_j \mu \circ S_j^{-1},$$

where  $S_j x = \rho x + b_j$ ,  $p_j > 0$  and  $\sum_{j=1}^{\ell} p_j = 1$ . Assume that  $\{S_j\}_{j=1}^{\ell}$  satisfies the finite type condition. Then we can construct a family (finite or countably infinite) of closed intervals  $I_j$  with disjoint interiors, such that  $\mu$  is supported on  $\bigcup_j I_j$  and the restricted measure  $\mu|_{I_j}$  of  $\mu$  on each interval  $I_j$  satisfies the complete multifractal formalism. Moreover the dimension spectrum  $\dim_H E_{\mu|_{I_j}}(\alpha)$  is independent of  $j$ .

For the global property of  $\mu$ , we remark that under the condition of Theorem 1.2, we do have  $\dim_H E_\mu(\alpha) = \inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\}$  for all  $\alpha \in [\tau'(+\infty), \tau'(0-)]$ , where  $\tau'(+\infty) := \lim_{q \rightarrow +\infty} \tau'(q)$ . This extends slightly the aforementioned result in [12, 31] (which says the formula holds for all  $\alpha = \tau'(t)$  with  $t > 0$ ). Furthermore if the equality  $\tau(\mu_k, q) = \tau(q)$  holds for all  $q < 0$  (for instance, this holds for the Erdős measure), then the complete multifractal formalism holds for  $\mu$ . For details, see Remark 5.3.

## 2. SOME BASIC LEMMAS

Let  $M(x)$  be a non-negative matrix-valued function defined on  $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$  satisfying the assumptions (H1) and (H2). For  $n \in \mathbb{N}$ , let  $\Sigma_n$  denote the set of all words of length  $n$  over  $\{1, \dots, m\}$ . Denote  $\Sigma^* = \bigcup_{n=1}^{\infty} \Sigma_n$ .

**Lemma 2.1.** *There exists a constant  $C > 0$  such that*

- (i) *for any  $I \in \Sigma_n, J \in \Sigma_\ell$ , there exists  $K \in \bigcup_{k=1}^r \Sigma_k$  such that  $\|M_{IKJ}\| \geq C\|M_I\| \|M_J\|$ .*
- (ii) *for any  $I \in \Sigma_n$  and  $\ell \in \mathbb{N}$ , there exists  $K \in \Sigma_\ell$  such that  $\|M_{IK}\| \geq C^\ell \|M_I\|$ .*
- (iii) *for any  $I \in \Sigma_n, J \in \Sigma_\ell$ , there exist  $1 \leq k \leq r, K_1 \in \Sigma_k$  and  $K_2 \in \Sigma_{2r-k}$  such that  $\|M_{IK_1JK_2}\| \geq C\|M_I\| \|M_J\|$ .*
- (iv) *for any  $I_1, I_2, \dots, I_n \in \Sigma_\ell$ , there exist  $K_1, K_2, \dots, K_{2n} \in \bigcup_{j=1}^{2r-1} \Sigma_j$  such that  $\sum_{j=1}^{2n} |K_j| = 2nr$  and*

$$(2.1) \quad C^n \prod_{j=1}^n \|M_{I_j}\| \leq \|M_{I_1(K_1 I_2 K_2)(K_3 I_3 K_4) \dots (K_{2n-3} I_n K_{2n-2}) K_{2n-1} K_{2n}}\| \leq C^{-n} \prod_{j=1}^n \|M_{I_j}\|.$$

**Proof.** Assume  $I \in \Sigma_n, J \in \Sigma_\ell$ . Since  $\sum_{k=1}^r H^k > \mathbf{0}$ , we have

$$\begin{aligned} \sum_{k=1}^r \sum_{K \in \Sigma_k} \|M_{IKJ}\| &= \left\| M_I \left( \sum_{k=1}^r H^k \right) M_J \right\| \\ &\geq D \|M_I\| \|M_J\| \end{aligned}$$

for some constant  $D > 0$ . It follows that there exists  $K \in \bigcup_{k=1}^r \Sigma_k$  such that

$$\|M_{IKJ}\| \geq \frac{D}{\#\{\bigcup_{k=1}^r \Sigma_k\}} \|M_I\| \|M_J\|,$$

where  $\#$  denotes the cardinality. This finishes the proof of part (i).

To prove part (ii) we can assume  $M_I \neq \mathbf{0}$ . By induction it suffices to prove part (ii) in the case  $\ell = 1$ . Choose  $j \in \{1, \dots, m\}$  so that  $M_j \neq \mathbf{0}$ . By (i), there exist  $1 \leq k \leq r$  and  $i_1 i_2 \dots i_k \in \Sigma_k$  such that

$$\|M_{I_{i_1 i_2 \dots i_k j}}\| \geq C_1 \|M_I\| \|M_j\| > 0$$

for some constant  $C_1 > 0$ . Since  $\|M_{I_{i_1}}\| \|M_{i_2 \dots i_k j}\| \geq \|M_{I_{i_1 i_2 \dots i_k j}}\|$ , we have

$$\|M_{I_{i_1}}\| \geq \frac{C_1 \|M_j\|}{\|M_{i_2 \dots i_k j}\|} \|M_I\|.$$

This completes the proof of part (ii).

Part (iii) follows immediately from part (i) and part (ii).

To see part (iv), using part (iii) inductively, we may find  $K_1, \dots, K_{2n-2}$  with  $|K_{2j-1}| + |K_{2j}| = 2r$  for  $1 \leq j \leq n-1$ , such that

$$\|M_{I_1(K_1 I_2 K_2)(K_3 I_3 K_4) \dots (K_{2n-3} I_n K_{2n-2})}\| \geq C^{n-1} \prod_{j=1}^n \|M_{I_j}\|.$$

Using part (ii) we may find  $K_{2n-1}K_{2n} \in \Sigma_{2r}$  such that the first inequality of (2.1) holds. Note that the second inequality also holds since

$$\begin{aligned} \|M_{I_1(K_1 I_2 K_2) \dots (K_{2n-3} I_n K_{2n-2}) K_{2n-1} K_{2n}}\| &\leq \left( \prod_{j=1}^n \|M_{I_j}\| \right) \left( \prod_{j=1}^{2n} \|M_{K_j}\| \right) \\ &\leq \left( \prod_{j=1}^n \|M_{I_j}\| \right) \left( \max_{1 \leq i \leq m} \|M_i\| \right)^{2nr}. \end{aligned}$$

This finishes the proof of the lemma.  $\square$

**Lemma 2.2.** *The limit in defining  $P_M(q)$  in (1.5) exists for any  $q \in \mathbb{R}$ . Moreover,  $P_M(q)$  is a convex function.*

**Proof.** Here we adopt a proof from [14]. Denote by  $s_n = \sum_{J \in \mathcal{N}_n} \|M_J\|^q$ .

We first consider the case  $q \geq 0$ . Since  $\|M_{IJ}\| \leq \|M_I\| \|M_J\|$  for any  $I \in \mathcal{N}_n$  and  $J \in \mathcal{N}_\ell$ , the sequence  $\{s_n\}$  is sub-multiplicative. Thus the limit  $\lim_{n \rightarrow \infty} \frac{\log s_n}{n}$  exists.

Now assume  $q < 0$ . By part (i) of Lemma 2.1, there is a constant  $C > 0$  such that for any  $I \in \mathcal{N}_n$  and  $J \in \mathcal{N}_\ell$ , there exists  $K \in \bigcup_{k=1}^r \Sigma_k$  satisfying

$$(2.2) \quad 0 < \|M_{IKJ}\| \leq C \|M_I\| \|M_J\|.$$

It implies  $s_n s_\ell \leq C^{-q} \sum_{k=1}^r s_{n+\ell+k}$ . From (2.2) we also deduce that for any  $I \in \Sigma^*$ , there exists  $i \in \Sigma_1$  such that  $M_{Ii} \neq 0$ ; Since  $\|M_{Ii}\| \leq C_1 \|M_I\|$  for some constant  $C_1 > 0$ , we have  $s_n \leq C_1^{-q} s_{n+1}$  for any integers  $n, \ell$ . It follows that  $s_n s_\ell \leq C' s_{n+\ell+r}$  for some constant  $C' > 0$  (depending on  $q$ ), which implies that  $a_n = \frac{1}{C'} s_{n-r}$  is super-multiplicative in the sense that  $a_{n+\ell} \geq a_n a_\ell$  for all  $n, \ell > r$ . Hence the limit  $\lim_{n \rightarrow \infty} \frac{\log s_n}{n}$  exists.

The convexity of  $P_M(q)$  follows by a standard argument.  $\square$

Let  $L_M$  and  $E_M(\cdot)$  be defined as in (1.2) and (1.3) respectively. It is clear that  $L_M = \{\alpha \in \mathbb{R} : E_M(\alpha) \neq \emptyset\}$ . Define

$$(2.3) \quad \alpha_M = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{I \in \mathcal{N}_n} \log \|M_I\|, \quad \beta_M = \lim_{n \rightarrow \infty} \frac{1}{n} \max_{I \in \mathcal{N}_n} \log \|M_I\|.$$

**Lemma 2.3.** (i) *The above limits in defining  $\alpha_M$  and  $\beta_M$  exist. Moreover,  $\alpha_M, \beta_M \in \mathbb{R}$ .*

(ii)  $L_M = [\alpha_M, \beta_M]$ .

(iii)  $\alpha_M = \lim_{q \rightarrow -\infty} \frac{P_M(q)}{q}$  and  $\beta_M = \lim_{q \rightarrow +\infty} \frac{P_M(q)}{q}$ .

**Proof.** Denote  $u_n = \max_{I \in \mathcal{N}_n} \|M_I\|$  and  $v_n = \min_{I \in \mathcal{N}_n} \|M_I\|$  for any  $n \in \mathbb{N}$ . It is clear that the sequence  $\{u_n\}$  is sub-multiplicative. Thus the limit in defining  $\beta_M$  exists. On the other hand by (2.2), there is a constant  $C > 0$  such that for any  $n, m \in \mathbb{N}$ , there exists  $k$  between 1 and  $r$  such that

$$(2.4) \quad v_{n+m+k} \leq C v_n v_m.$$

Using (2.2) again we deduce that for any  $I \in \mathcal{N}_n$  there exists  $i \in \Sigma_1$  such that  $\|M_{Ii}\| \neq 0$ . Since  $\|M_{Ii}\| \leq C_1 \|M_I\|$  for some constants  $C_1 > 0$ , we have  $v_{n+1} \leq C_1 v_n$ . This fact together with (2.4) yields  $v_{n+m+r} \leq C' v_n v_m$  for some constants  $C' > 0$ . It implies that the sequence  $\{C' v_{n-r}\}_n$  is sub-multiplicative. Thus the limit in defining  $\alpha_M$  exists. To see  $\alpha_M, \beta_M \in \mathbb{R}$ , we only need to observe that

$$t^n \leq v_n \leq u_n \leq \left( \max_{1 \leq i \leq m} \|M_i\| \right)^n,$$

where  $t$  is the smallest non-zero entry appearing in the matrices  $M_i$ ,  $1 \leq i \leq m$ . This finishes the proof of part (i).

To see (ii), by the definitions of  $L_M$ ,  $\alpha_M$  and  $\beta_M$  we have  $L_M \subseteq [\alpha_M, \beta_M]$  immediately. Thus to complete the proof of part (ii), we only need to show that for any  $s \in [\alpha_M, \beta_M]$ , there exists  $y \in \Sigma$  such that  $\lambda_M(y) = s$ . Now fix  $s \in [\alpha_M, \beta_M]$  and write  $s = p\alpha_M + (1-p)\beta_M$  for some  $p \in [0, 1]$ . For  $n \in \mathbb{N}$ , construct a sequence of words  $\{\omega_n\}_{n=1}^\infty$  such that  $\omega_n \in \mathcal{N}_n$  and

$$\|M_{\omega_{2n}}\| = u_{2n}, \quad \|M_{\omega_{2n-1}}\| = v_{2n-1}.$$

Construct a sequence of positive integers  $\{N_n\}$  by

$$N_n = \begin{cases} \lfloor pn + \log n \rfloor, & \text{if } n \text{ is odd,} \\ \lfloor (1-p)n + \log n \rfloor, & \text{otherwise,} \end{cases}$$

where  $\lfloor x \rfloor$  denotes the integral part of  $x$ . It can be checked directly that

$$(2.5) \quad \lim_{n \rightarrow \infty} N_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{nN_n}{\sum_{i=1}^n iN_i} = 0, \quad \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n (2i-1)N_{2i-1}}{\sum_{j=1}^{2n} jN_j} = p.$$

Now define a sequence of words in the following way:

$$\underbrace{\omega_1, \dots, \omega_1}_{N_1}, \underbrace{\omega_2, \dots, \omega_2}_{N_2}, \dots, \underbrace{\omega_n, \dots, \omega_n}_{N_n}, \dots$$

We label this sequence as  $\{\nu_n\}$ . Using Lemma 2.1 (i) inductively, we can find a sequence of words  $\{K_n\}$  such that  $K_n \in \bigcup_{k=1}^r \Sigma_k$  and

$$\|M_{\nu_1 K_1 \nu_2 K_2 \dots \nu_\ell K_\ell}\| \geq C^\ell \prod_{k=1}^{\ell} \|M_{\nu_k}\|, \quad \forall \ell \in \mathbb{N},$$

where  $C$  is the constant in Lemma 2.1. Note that the left-hand side of the above inequality is not greater than  $D^{r\ell} \prod_{k=1}^{\ell} \|M_{\nu_k}\|$ , where  $D = \max\{1, \max_{1 \leq i \leq m} \|M_i\|\}$ . Define  $y \in \Sigma$  by

$$y = \nu_1 K_1 \nu_2 K_2 \dots \nu_\ell K_\ell \dots$$

One can check directly that  $\lambda_M(y) = s$  by using (2.5) and the above two inequalities about  $\|M_{\nu_1 K_1 \nu_2 K_2 \dots \nu_\ell K_\ell}\|$ . This finishes the proof of (ii).

To see (iii), note that for each  $n \in \mathbb{N}$ ,

$$\begin{cases} (u_n)^q \leq \sum_{I \in \mathcal{N}_n} \|M_I\|^q \leq m^n (u_n)^q, & \forall q \geq 0 \\ (v_n)^q \leq \sum_{I \in \mathcal{N}_n} \|M_I\|^q \leq m^n (v_n)^q, & \forall q < 0 \end{cases}$$

which implies that

$$(2.6) \quad \begin{cases} q\beta_M \leq P_M(q) \leq \log m + q\beta_M, & \forall q \geq 0 \\ q\alpha_M \leq P_M(q) \leq \log m + q\alpha_M, & \forall q < 0 \end{cases}$$

By taking the limits we obtain the desired result.  $\square$

For any  $\alpha \in \mathbb{R}$ ,  $n \in \mathbb{N}$  and  $\epsilon > 0$ , define

$$F(\alpha; n, \epsilon) = \left\{ I \in \Sigma_n : \left| \frac{1}{n} \log \|M_I\| - \alpha \right| \leq \epsilon \right\}$$

and  $f(\alpha; n, \epsilon) = \#F(\alpha; n, \epsilon)$ . Moreover for  $\ell \in \mathbb{N}$ , define

$$F_\ell(\alpha; n\ell, \epsilon) = \left\{ i_1 \dots i_{n\ell} \in \Sigma_{n\ell} : \left| \frac{1}{n\ell} \log \prod_{j=1}^n \|M_{i_{(j-1)\ell+1} i_{(j-1)\ell+2} \dots i_{j\ell}}\| - \alpha \right| \leq \epsilon \right\}$$

and  $f_\ell(\alpha; n\ell, \epsilon) = \#F_\ell(\alpha; n\ell, \epsilon)$ .

**Lemma 2.4.** *For any  $\alpha \in \mathbb{R}$ ,  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $n, \ell \in \mathbb{N}$  satisfy  $\ell/n < \delta$ , then  $f(\alpha; n + \ell, 2\epsilon) \geq f(\alpha; n, \epsilon)$ .*

**Proof.** By Lemma 2.1, there is a constant  $C > 0$  such that for any  $I \in F(\alpha; n, \epsilon)$  and  $\ell \in \mathbb{N}$ , there exists  $K \in \Sigma_\ell$  satisfying  $\|M_{IK}\| \geq C^\ell \|M_I\|$ . Meanwhile it is obvious that  $\|M_{IK}\| \leq C_1^\ell \|M_I\|$ , where  $C_1 = \max_{1 \leq i \leq m} \|M_i\|$ . By these two inequalities, we have

$$\frac{1}{n + \ell} (\ell \log C + \log \|M_I\|) \leq \frac{1}{n + \ell} \log \|M_{IK}\| \leq \frac{1}{n + \ell} (\ell \log C_1 + \log \|M_I\|).$$

Thus there exists  $\delta > 0$  such that if  $\ell/n < \delta$  then

$$\left| \frac{\log \|M_{IK}\|}{n + \ell} - \alpha \right| < 2\epsilon.$$

This implies  $f(\alpha; n + \ell, 2\epsilon) \geq f(\alpha; n, \epsilon)$ .  $\square$

**Lemma 2.5.** *There is a constant  $D > 0$  such that for any  $\epsilon > 0$ , there exists  $N > 0$  (depending on  $\epsilon$ ) such that for any  $\alpha \in L_M$ ,*

$$(2.7) \quad f(\alpha; p(\ell + 2r), 2\epsilon) \geq D^p f_\ell(\alpha; p\ell, \epsilon) \geq D^p (f(\alpha; \ell, \epsilon))^p, \quad \forall p \in \mathbb{N}, \forall \ell > N.$$

**Proof.** It is obvious that for any  $I_1, \dots, I_p \in F(\alpha; \ell, \epsilon)$ ,  $I_1 \dots I_p \in F_\ell(\alpha; p\ell, \epsilon)$ . It implies that  $f_\ell(\alpha; p\ell; \epsilon) \geq (f(\alpha; \ell; \epsilon))^p$ . Thus the second inequality in (2.7) always holds.

To prove the first inequality, let  $I_1, \dots, I_p \in \Sigma_\ell$  be any words with  $I_1 \dots I_p \in F_\ell(\alpha; p\ell, \epsilon)$ . By part (iv) of Lemma 2.1, there exist  $K_1, \dots, K_{2p} \in \bigcup_{j=1}^{2r-1} \Sigma_j$  such that  $\sum_{j=1}^{2p} |K_j| = 2pr$  and

$$C^p \prod_{j=1}^p \|M_{I_j}\| \leq \|M_{I_1(K_1 I_2 K_2)(K_3 I_3 K_4) \dots (K_{2p-3} I_p K_{2p-2}) K_{2p-1} K_{2p}}\| \leq C^{-p} \prod_{j=1}^p \|M_{I_j}\|,$$

where  $C$  is a constant. Hence if  $\ell > \epsilon^{-1}(|\log C| + 2r \max\{|\beta| : \beta \in L_M\})$ , then

$$I_1(K_1 I_2 K_2)(K_3 I_3 K_4) \dots (K_{2p-3} I_p K_{2p-2}) K_{2p-1} K_{2p} \in F(\alpha; p(\ell + 2r), 2\epsilon).$$

Note that when  $I_1 \dots I_p$  is taken over the set  $F_\ell(\alpha; p\ell, \epsilon)$ , by the pigeon hole principle, we get at least

$$\left( \# \left( \bigcup_{j=1}^{2r-1} \Sigma_j \right) \right)^{-2p} f_\ell(\alpha; p\ell, \epsilon)$$

many different elements of form  $I_1(K_1 I_2 K_2)(K_3 I_3 K_4) \dots (K_{2p-3} I_p K_{2p-2}) K_{2p-1} K_{2p}$  which belong to  $F(\alpha; p(\ell + 2r), 2\epsilon)$ . Here  $\left( \# \left( \bigcup_{j=1}^{2r-1} \Sigma_j \right) \right)^{2p}$  is an upper bound of the number of all possible different elements  $(K_1, \dots, K_{2p})$ . Taking  $D = \left( \# \left( \bigcup_{j=1}^{2r-1} \Sigma_j \right) \right)^{-2}$ , we have  $f(\alpha; p(\ell + 2r), 2\epsilon) \geq D^p f_\ell(\alpha; p\ell, \epsilon)$ . This finishes the proof of the lemma.  $\square$

Fix  $\ell \in \mathbb{N}$ . Define

$$(2.8) \quad E_M^{(\ell)}(\alpha) = \left\{ (x_i)_{i=1}^\infty \in \Sigma : \lim_{n \rightarrow \infty} \frac{1}{n\ell} \log \left( \prod_{j=1}^n \|M_{x_{(j-1)\ell+1} \dots x_{j\ell}}\| \right) = \alpha \right\}, \quad \alpha \in \mathbb{R}$$

and

$$(2.9) \quad P_M(\ell, q) = \frac{1}{\ell} \log (\sum_{J \in \mathcal{N}_\ell} \|M_J\|^q), \quad q \in \mathbb{R}.$$

**Proposition 2.6.** *If  $\alpha = P'_M(\ell, q)$  for some  $q \in \mathbb{R}$ , then  $E_M^{(\ell)}(\alpha) \neq \emptyset$  and*

$$(2.10) \quad \dim_H E_M^{(\ell)}(\alpha) = \frac{1}{\log m} (P_M(\ell, q) - \alpha q).$$

The proof of the above result is based on the following theorem of Billingsley:

**Proposition 2.7.** *(c.f. [19], P. 99, see also [5, 50]). Let  $(X, d)$  be a metric space ( $\mathbb{R}^n$  or  $\Sigma$ ). Let  $\mu$  be a Borel probability measure on  $X$ . For a Borel set  $E \subset X$ , we have  $a \leq \dim_H E \leq b$  if*

$$\mu(E) > 0, \quad E \subset \left\{ x \in X : a \leq \liminf_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r} \leq b \right\}.$$

**Proof of Proposition 2.6.** Assume  $\alpha = P'_M(\ell, q)$  for some  $q \in \mathbb{R}$ . Then

$$\alpha = \frac{1}{\ell} e^{-\ell P_M(\ell, q)} \sum_{J \in \mathcal{N}_\ell} \|M_J\|^q \log \|M_J\|.$$

Define a product-like measure  $\mu$  on  $\Sigma$  such that

$$(2.11) \quad \mu([x_1 x_2 \dots x_{n\ell}]) = e^{-n\ell P_M(\ell, q)} \prod_{j=1}^n \|M_{x_{(j-1)\ell+1} \dots x_{j\ell}}\|^q$$

for any  $n$  and any cylinder  $[x_1 x_2 \dots x_{n\ell}]$  with  $x_{(j-1)\ell+1} \dots x_{j\ell} \in \mathcal{N}_\ell$  for each  $1 \leq j \leq n$ . It is clear  $\mu$  is  $\sigma^\ell$ -invariant and  $\sigma^\ell$ -ergodic. By the Birkhoff's ergodic theorem, we have for  $\mu$  a.e.  $x = (x_i)_{i=1}^\infty \in \Sigma$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log \mu([x_1 x_2 \dots x_{n\ell}])}{\log m^{-n\ell}} &= \lim_{n \rightarrow \infty} \frac{-n\ell P_M(\ell, q) + \sum_{j=1}^n q \log \|M_{x_{(j-1)\ell+1} \dots x_{j\ell}}\|}{-n\ell \log m} \\ &= \frac{1}{\log m} \left( P_M(\ell, q) - q \ell^{-1} e^{-\ell P_M(\ell, q)} \sum_{J \in \mathcal{N}_\ell} \|M_J\|^q \log \|M_J\| \right) \\ &= \frac{1}{\log m} (P_M(\ell, q) - \alpha q). \end{aligned}$$

Note also that

$$(x_i)_{i=1}^\infty \in E_M^{(\ell)}(\alpha) \iff \lim_{n \rightarrow \infty} \frac{\log \mu([x_1 x_2 \dots x_{n\ell}])}{\log m^{-n\ell}} = \frac{1}{\log m} (P_M(\ell, q) - \alpha q).$$

This proves the proposition by using the Billingsley theorem.  $\square$

**Proposition 2.8.** *Let  $\{M_i : i = 1, \dots, m\}$  be a family of  $d \times d$  complex matrices which is irreducible in the following sense: there is no proper linear subspace  $V$  of  $\mathbb{C}^d$  such that  $M_i V \subseteq V$  for all  $1 \leq i \leq m$ . Then there exist  $C > 0$  and  $k \in \mathbb{N}$  such that for any words  $I, J \in \Sigma^* = \bigcup_{n=1}^\infty \{1, \dots, m\}^n$ , there exists a word  $K \in \bigcup_{n=1}^k \{1, \dots, m\}^n$  such that*

$$\|M_{IKJ}\| \geq C \|M_I\| \|M_J\|,$$

where  $\|\cdot\|$  denotes the standard matrix norm.

*Proof.* We prove the proposition by contradiction. Assume that the proposition is not true. Then for any  $\delta > 0$  and  $n \in \mathbb{N}$ , there exist  $I, J \in \Sigma^*$  such that  $M_I, M_J \neq \mathbf{0}$  and

$$\|M_{IKJ}\| \leq \delta \|M_I\| \|M_J\|, \quad \forall K \in \Sigma^* \text{ with } |K| \leq n,$$

where  $|K|$  denotes the length of  $K$ . Let  $u_I$  be a unit row vector and  $v_J$  a unit column vector in  $\mathbb{C}^d$  such that

$$\|M_I\| = \|u_I M_I\|, \quad \|M_J\| = \|M_J v_J\|.$$

Define  $\widetilde{u}_I = \frac{u_I M_I}{\|M_I\|}$  and  $\widetilde{v}_J = \frac{M_J v_J}{\|M_J\|}$ . Then

$$|\widetilde{u}_I M_K \widetilde{v}_J| \leq \delta, \quad \forall K \in \Sigma^* \text{ with } |K| \leq n.$$

Take two sequences  $\{\delta_\ell\} \downarrow 0$  and  $\{n_\ell\} \uparrow \infty$ . Then for each  $\ell$ , there are unit column vectors  $w_{\ell,1}$  and  $w_{\ell,2}$  such that

$$|w_{\ell,1}^t M_K w_{\ell,2}| \leq \delta_\ell, \quad \forall K \in \Sigma^* \text{ with } |K| \leq n_\ell.$$

Let  $(w_1, w_2)$  be a limit point of  $\{(w_{\ell,1}, w_{\ell,2})\}_{\ell \in \mathbb{N}}$ . Then we have

$$|w_1^t M_K w_2| = 0, \quad \forall K \in \Sigma^*.$$

Let  $V$  be the linear space spanned by  $\{M_K w_2 : K \in \Sigma^*\}$ . It is clear that  $M_i V \subseteq V$  for all  $1 \leq i \leq m$ . However  $V$  is perpendicular to  $w_1$ , which means that  $V$  is a proper subspace of  $\mathbb{C}^d$ . This contradicts the irreducibility of  $\{M_i : 1 \leq i \leq m\}$ .  $\square$

We remark that the above result and its extension for singular value functions for matrix products were independently obtained by Falconer and Sloan [22]. They also obtained partial results on the multifractal analysis of norms of matrix products [23].

### 3. A DIMENSION FORMULA FOR $E_M(\alpha)$

In this section we give a dimensional formula for the set  $E_M(\alpha)$ .

**Proposition 3.1.** *For  $\alpha \in L_M$ , we have*

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log f(\alpha; n, \epsilon)}{\log m^n} = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log f(\alpha; n, \epsilon)}{\log m^n} \quad (=:\Lambda_M(\alpha)).$$

The function  $\Lambda_M : L_M \rightarrow [0, 1]$  is concave and continuous.

**Proof.** The proposition follows from Lemma 2.4 and Lemma 2.5 by a standard argument. For a similar approach, see the proof of Proposition 3.2 in [11].  $\square$

**Proposition 3.2.** *For  $\alpha \in L_M$ , we have*

$$\dim_H E_M(\alpha) = \dim_P E_M(\alpha) = \Lambda_M(\alpha).$$

Furthermore, for any word  $\xi \in \Sigma^*$  with  $M_\xi \neq \mathbf{0}$ , there exists  $\Gamma = \Gamma(\xi) \subset E_M(\alpha)$  such that  $\dim_H \Gamma = \Lambda_M(\alpha)$  and for any  $x = (x_j)_{j=1}^\infty \in \Gamma$ , there exists a sequence of positive integers  $\{n_k(x)\}_{k=1}^\infty \uparrow \infty$  such that  $n_1(x) = 1$ ,  $\lim_{k \rightarrow \infty} \frac{n_{k+1}(x)}{n_k(x)} = 1$ , and  $\xi$  appears in  $(x_j)_{j=1}^\infty$  at each position  $n_k(x)$  (here we say  $\xi = u_1 \dots u_s$  appears in  $(x_j)_{j=1}^\infty$  at position  $n$  if  $x_n x_{n+1} \dots x_{n+s-1} = \xi$ ).

**Proof.** We divide the proof into two small steps.

*Step 1.*  $\dim_P E_M(\alpha) \leq \Lambda_M(\alpha)$ ,  $\forall \alpha \in L_M$ .

The proof of this step is standard. Let

$$G(\alpha; k, \epsilon) = \bigcap_{n=k}^\infty \left\{ x = (x_i)_{i=1}^\infty \in \Sigma : \left| \frac{1}{n} \log \|M_{x_1} \cdots M_{x_n}\| - \alpha \right| < \epsilon \right\}.$$

It is clear that for any  $\epsilon > 0$ ,

$$E_M(\alpha) \subset \bigcup_{k=1}^\infty G(\alpha; k, \epsilon).$$

Notice that if  $n \geq k$ ,  $G(\alpha; k, \epsilon)$  is covered by the union of all cylinders  $[\omega]$  with  $\omega \in F(\alpha; n, \epsilon)$  whose total number is  $f(\alpha; n, \epsilon)$ . Therefore we have the following estimate

$$\overline{\dim}_B G(\alpha; k, \epsilon) \leq \limsup_{n \rightarrow \infty} \frac{\log f(\alpha; n, \epsilon)}{\log m^n} \quad (\forall \epsilon > 0, \forall k \geq 1),$$

where  $\overline{\dim}_B$  denotes the upper box dimension (see [20]). On the other hand, by using the  $\sigma$ -stability of the packing dimension, we have

$$\begin{aligned} \dim_P E_M(\alpha) &\leq \dim_P \left( \bigcup_{k=1}^{\infty} G(\alpha; k, \epsilon) \right) \leq \sup_k \dim_P G(\alpha; k, \epsilon) \\ &\leq \sup_k \overline{\dim}_B G(\alpha; k, \epsilon) \leq \limsup_{n \rightarrow \infty} \frac{\log f(\alpha; n, \epsilon)}{\log m^n}. \end{aligned}$$

Taking  $\epsilon \rightarrow 0$ , we obtain the desired result.

*Step 2.* For any given  $\xi \in \Sigma^*$  with  $M_\xi \neq \mathbf{0}$  and  $\delta > 0$ , there exists  $\Gamma \subset E_M(\alpha)$  such that  $\dim_H \Gamma > \Lambda_M(\alpha) - \delta$  and for any  $x = (x_j)_{j \geq 1} \in \Gamma$ , there exists a sequence of positive integers  $\{n_k(x)\}_{k=1}^{\infty} \uparrow \infty$  such that  $n_1(x) = 1$ ,  $\lim_{k \rightarrow \infty} \frac{n_{k+1}(x)}{n_k(x)} = 1$  and  $\xi$  appears in  $(x_j)_{j \geq 1}$  at each position  $n_k(x)$ .

We first construct for any index  $(i, j) \in \{1, 2, \dots, d\}^2$ , a word  $\eta(i, j) \in \Sigma^*$  such that  $\xi$  appears in  $\eta(i, j)$  as a subword and the  $(i, j)$ -th entry of  $M_{\eta(i, j)}$  is positive. To achieve this purpose, we fix an index  $(s, t)$  such that the  $(s, t)$ -th entry of  $M_\xi$  is positive. By the assumption (H2), for any  $1 \leq i, j \leq d$ , there exist words  $I_1, I_2 \in \Sigma^*$  such that the  $(i, s)$ -th entry of  $M_{I_1}$  and the  $(t, j)$ -th entry of  $M_{I_2}$  are positive. Set  $\eta(i, j) = I_1 \xi I_2$ . Then  $\eta(i, j)$  satisfies the desired property.

By Proposition 3.1, there are  $\ell_j \uparrow \infty$  and  $\epsilon_j \downarrow 0$  such that

$$f(\alpha; \ell_j, \epsilon_j) > m^{\ell_j(\Lambda_M(\alpha) - \frac{\delta}{2})}.$$

Define a new sequence  $\{\ell_j^*\}$  in the following manner

$$\underbrace{\ell_1, \dots, \ell_1}_{N_1}; \underbrace{\ell_2, \dots, \ell_2}_{N_2}; \dots; \underbrace{\ell_j, \dots, \ell_j}_{N_j}; \dots$$

where  $N_j$  is defined recursively by

$$N_j = 2^{\ell_{j+1} + N_{j-1}} \quad (j \geq 2); \quad N_1 = 1.$$

Set  $\epsilon_j^* = \epsilon_i$  when  $\ell_j^* = \ell_i$ . By the pigeon hole principle, for each  $j$ , there exist an index  $(s_j, t_j)$  and a subclass  $G_j$  of  $F(\alpha; \ell_j^*, \epsilon_j^*)$  such that  $\#G_j \geq d^{-2} f(\alpha; \ell_j^*, \epsilon_j^*)$  and for any  $I \in G_j$ , the  $(s_j, t_j)$ -th entry of  $M_I$  is greater than or equal to  $d^{-2} \|M_I\|$ .

Define

$$\Theta^* = \prod_{j=1}^{\infty} G_j.$$

Observe that  $\Theta^*$  is a homogeneous Moran set in  $\Sigma$  (see, e.g., [11]). More precisely  $\Theta^*$  is constructed as follows. At level 0, we have only the initial cylinder  $\Sigma$ . In step  $j$ , cut a cylinder of level  $j-1$  into

$m^{\ell_j^*}$  cylinders and pick up  $\#G_j$  ones. Thus we have (see, e.g., [11, Proposition 3.1])

$$\begin{aligned} \dim_H \Theta^* &\geq \liminf_{k \rightarrow \infty} \frac{\log \prod_{j=1}^k \#G_j}{\log \prod_{j=1}^{k+1} m^{\ell_j^*}} \\ &\geq \liminf_{k \rightarrow \infty} \frac{\log \prod_{j=1}^k (d^{-2} f(\alpha; \ell_j^*, \epsilon_j^*))}{\log \prod_{j=1}^{k+1} m^{\ell_j^*}} \\ &\geq \Lambda_M(\alpha) - \delta. \end{aligned}$$

Now for any  $y^* = I_1 I_2 \dots \in \Theta^*$  with  $I_j \in G_j$ , we define

$$y^{**} = \xi \eta(t, s_1) I_1 \eta(t_1, s_2) I_2 \eta(t_2, s_3) \dots I_j \eta(t_j, s_{j+1}) \dots$$

Define  $\Theta^{**} \subset \Sigma$  as the set of sequences  $y^{**}$ . By considering the map  $y^* \mapsto y^{**}$  which is nearly bi-Lipschitz, we have  $\dim_H \Theta^{**} = \dim_H \Theta^*$ . On the other hand a direct check shows  $\Theta^{**} \subset E_M(\alpha)$ . Set  $\Gamma = \Theta^{**}$ . Then  $\Gamma$  satisfies the desired property. This finishes the proof of the proposition.  $\square$

#### 4. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we need the following

**Proposition 4.1.** [11, Propostion 2.5]. *Let  $f$  be a convex real-valued function on  $\mathbb{R}$ . Denote*

$$(4.1) \quad a = \lim_{x \rightarrow -\infty} \frac{f(x)}{x}, \quad b = \lim_{x \rightarrow \infty} \frac{f(x)}{x}.$$

- (i) *Suppose that  $\{f_n\}_{n=1}^\infty$  is a sequence of differentiable convex functions converging to  $f$  pointwise. Then for any  $c \in (a, b)$ , there exist  $N > 0$  and a uniformly bounded sequence of real numbers  $\{x_n\}_{n \geq N}$  such that  $f'_n(x_n) = c$ .*
- (ii) *Assume  $-\infty < a < b < \infty$ . Then we have*

$$\overline{\lim}_{z \uparrow b} \inf_{x \in \mathbb{R}} \{-zx + f(x)\} \geq \inf_{x \in \mathbb{R}} \{-bx + f(x)\}, \quad \overline{\lim}_{z \downarrow a} \inf_{x \in \mathbb{R}} \{-zx + f(x)\} \geq \inf_{x \in \mathbb{R}} \{-ax + f(x)\}.$$

**Proof of Theorem 1.1.** We divide the proof into 2 steps:

$$\text{Step 1. } \dim_P E_M(\alpha) \leq \frac{1}{\log m} (-\alpha q + P_M(q)) \quad \forall \alpha \in L_M, \forall q \in \mathbb{R}.$$

For any  $\alpha \in L_M$ ,  $\epsilon > 0$  and  $n \in \mathbb{N}$ , let  $f(\alpha; n, \epsilon)$  be defined as in Section 2. Then

$$\sum_{I \in \mathcal{N}_n} \|M_I\|^q \geq \begin{cases} f(\alpha; n, \epsilon) \exp(nq(\alpha - \epsilon)), & \text{if } q \geq 0, \\ f(\alpha; n, \epsilon) \exp(nq(\alpha + \epsilon)), & \text{if } q < 0, \end{cases}$$

which implies that for any  $q \in \mathbb{R}$ ,

$$P_M(q) \geq q\alpha + \lim_{\epsilon \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{\log f(\alpha; n, \epsilon)}{n}.$$

Applying Propositions 3.1 and 3.2 gives

$$P_M(q) \geq \alpha q + (\log m) \dim_P E_M(\alpha).$$

*Step 2. For all  $\alpha \in L_M$ ,*

$$(4.2) \quad \dim_H E_M(\alpha) \geq \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha q + P_M(q)\}.$$

To show the above inequality, we at first consider a trivial case:  $\alpha_M = \beta_M$ , where  $\alpha_M$  and  $\beta_M$  are defined as in (2.3). In this case,  $\lambda_M(x) = \alpha_M$  for all  $x \in \Sigma$ . Thus by (2.6), we have

$$\dim_H E_M(\alpha_M) = \dim_H \Sigma = 1 \geq \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha_M q + P_M(q)\}.$$

In the following part we assume  $\alpha_M \neq \beta_M$ .

First we consider  $\alpha \in (\alpha_M, \beta_M)$ . The case  $\alpha = \alpha_M$  or  $\beta_M$  will be treated later. Let  $P_M(\ell, q)$  and  $E_M^{(\ell)}(\alpha)$  be defined as in (2.9) and (2.8) respectively. By Lemma 2.2, the sequence  $\{P_M(\ell, q)\}_{\ell=1}^{\infty}$  converges to  $P_M(q)$  pointwisely. Thus by Proposition 4.1, there exists  $N > 0$  such that there is a uniformly bounded sequence of real number  $\{q_\ell\}_{\ell \geq N}$  such that  $P_M'(\ell, q_\ell) = \alpha$ . By Proposition 2.6,

$$\dim_H E_M^{(\ell)}(\alpha) = P_M(\ell, q_\ell) - \alpha q_\ell, \quad \forall \ell \geq N.$$

Note that for any  $\epsilon > 0$  and  $n \in \mathbb{N}$ , the collection  $\{[J] : J \in F_\ell(\alpha; n\ell, \epsilon)\}$  is a  $m^{-n\ell}$ -cover of  $E_M^{(\ell)}(\alpha)$ . This implies

$$\liminf_{n \rightarrow \infty} \frac{\log f_\ell(\alpha; n\ell, \epsilon)}{n\ell \log m} \geq \dim_H E_M^{(\ell)}(\alpha) = P_M(\ell, q_\ell) - \alpha q_\ell.$$

Combining it with (2.7) yields

$$\liminf_{n \rightarrow \infty} \frac{\log f(\alpha; n\ell + 2nr, 2\epsilon)}{(n\ell + 2nr) \log m} \geq \frac{\ell}{\ell + 2r} \cdot \frac{1}{\log m} (P_M(\ell, q_\ell) - \alpha q_\ell), \quad \forall \epsilon > 0.$$

Hence by Proposition 3.2, we have

$$(4.3) \quad \dim_H E_M(\alpha) \geq \frac{\ell}{\ell + 2r} \cdot \frac{1}{\log m} (P_M(\ell, q_\ell) - \alpha q_\ell), \quad \forall \ell \geq N.$$

Since the sequence  $\{q_\ell\}_{\ell \geq N}$  is uniformly bounded, there exists a subsequence  $\{q_{\ell_i}\}$  which converges to a finite point  $q_\infty$ . We are going to show

$$(4.4) \quad \lim_{i \rightarrow \infty} (P_M(\ell_i, q_{\ell_i}) - \alpha q_{\ell_i}) = P_M(q_\infty) - \alpha q_\infty.$$

Denote  $t = \max\{\|M_i\| : 1 \leq i \leq m\}$ , and define a new matrix function  $\tilde{M} := \frac{1}{t}M$ . It is easily checked that

$$(4.5) \quad P_M(\ell, q) = q \log t + P_{\tilde{M}}(\ell, q), \quad P_M(q) = q \log t + P_{\tilde{M}}(q).$$

Since  $\|\tilde{M}_i\| \leq 1$  for all  $1 \leq i \leq m$ ,  $P_{\tilde{M}}(\ell, q)$  and  $P_{\tilde{M}}(q)$  are monotone decreasing functions of  $q$ . Now for any  $\epsilon > 0$ , we can find  $a, b \in \mathbb{R}$  with  $a < q_\infty < b$  such that

$$P_{\tilde{M}}(a) < P_{\tilde{M}}(q_\infty) + \epsilon, \quad P_{\tilde{M}}(b) > P_{\tilde{M}}(q_\infty) - \epsilon.$$

When  $i$  is large enough,  $q_{\ell_i} \in (a, b)$ , and thus

$$P_{\tilde{M}}(\ell_i, b) \leq P_{\tilde{M}}(\ell_i, q_{\ell_i}) \leq P_{\tilde{M}}(\ell_i, a).$$

It follows that

$$\limsup_{i \rightarrow \infty} P_{\tilde{M}}(\ell_i, q_{\ell_i}) \leq \lim_{i \rightarrow \infty} P_{\tilde{M}}(\ell_i, a) = P_{\tilde{M}}(a) < P_{\tilde{M}}(q_\infty) + \epsilon.$$

and

$$\liminf_{i \rightarrow \infty} P_{\tilde{M}}(\ell_i, q_{\ell_i}) \geq \lim_{i \rightarrow \infty} P_{\tilde{M}}(\ell_i, b) = P_{\tilde{M}}(b) > P_{\tilde{M}}(q_\infty) - \epsilon.$$

Since  $\epsilon > 0$  is arbitrary, we have

$$\lim_{i \rightarrow \infty} P_{\tilde{M}}(\ell_i, q_{\ell_i}) = P_{\tilde{M}}(q_\infty).$$

Combining it with (4.5), we obtain  $\lim_{i \rightarrow \infty} P_M(\ell_i, q_{\ell_i}) = P_M(q_\infty)$ , which implies (4.4). Thus by (4.3), we have

$$\dim_H E_M(\alpha) \geq \frac{1}{\log m} (P_M(q_\infty) - \alpha q_\infty).$$

This establishes (4.2).

Now we consider the case  $\alpha = \alpha_M$  (the case  $\alpha = \beta_M$  can be treated in an identical way). By Proposition 3.2 and 3.1, the function  $\dim_H E_M(x)$  of  $x$  is continuous on  $L_M$ . Consequently,

$$\dim_H E_M(\alpha_M) = \lim_{z \downarrow \alpha_M} \dim_H E_M(z),$$

and thus

$$\dim_H E_M(\alpha_M) \geq \frac{1}{\log m} \lim_{z \downarrow \alpha_M} \inf_{q \in \mathbb{R}} \{-zq + P_M(q)\}.$$

By Proposition 4.1 (ii), we have

$$\dim_H E_M(\alpha_M) \geq \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha_M q + P_M(q)\},$$

which finishes the proof of (4.2).  $\square$

## 5. PROOF OF THEOREM 1.2

Let  $S_j x = \rho x + b_j$  ( $j = 1, 2, \dots, \ell$ ) satisfy the finite type condition. That is, there exists a finite set  $\Gamma$  of non-negative numbers such that for each integer  $n \in \mathbb{N}$  and any two indices  $J = j_1 \dots j_n$  and  $J' = j'_1 \dots j'_n$ ,

$$(5.1) \quad \text{either} \quad \rho^{-n} |S_J(0) - S_{J'}(0)| > c \quad \text{or} \quad \rho^{-n} |S_J(0) - S_{J'}(0)| \in \Gamma,$$

where  $S_J$  denotes the composition  $S_{j_1} \circ \dots \circ S_{j_n}$  and  $c = (1 - \rho)^{-1} (\max_{1 \leq j \leq \ell} b_j - \min_{1 \leq i \leq \ell} b_i)$ . Let  $\mu$  be the self-similar measure generated by  $\{S_j\}_{j=1}^\ell$  with the probability weight  $\{p_j\}_{j=1}^\ell$ , i.e.,  $\mu$  is the unique Borel probability measure on  $\mathbb{R}$  satisfying the following relation:

$$(5.2) \quad \mu = \sum_{j=1}^{\ell} p_j \mu \circ S_j^{-1}.$$

Let  $K$  denote the self-similar set generated by  $\{S_j\}_{j=1}^\ell$ , i.e.,  $K$  is the unique non-empty compact set satisfying  $K = \bigcup_{j=1}^{\ell} S_j(K)$ .

The following proposition describes the local structure of  $\mu$ :

**Proposition 5.1.** *Let the family  $\{S_j\}_{j=1}^\ell$  and the self-similar measure  $\mu$  be given as above. Then there exist two positive integers  $m$  and  $d$ , an  $m \times m$  irreducible 0-1 matrix  $A = (A_{ij})_{1 \leq i, j \leq m}$ ,  $m \times d$  non-negative matrices  $M_i$  ( $1 \leq i \leq m$ ), such that*

- (a)  $\sum_{j=1}^m M_j$  is irreducible.
- (b)  $M_{j_1} M_{j_2} \dots M_{j_n} \neq \mathbf{0}$  if and only if the word  $J = j_1 j_2 \dots j_n$  is  $A$ -admissible (i.e.,  $A_{j_k j_{k+1}} = 1$  for  $k = 1, \dots, n-1$ ).
- (c) Let  $\Sigma_A$  denote the set  $\{(x_i)_{i=1}^\infty \in \Sigma : A_{x_i x_{i+1}} = 1 \text{ for all } i \geq 1\}$ . Then there exists a family (finite or countably infinite) of closed intervals  $\{I_k\}_{k \in \Lambda}$  with disjoint interiors such that the endpoints of  $I_k$  belong to  $K$  for all  $k \in \Lambda$ ,  $\mu(\bigcup_{k \in \Lambda} I_k) = 1$ , and for each  $k \in \Lambda$  there exists a surjective map  $\pi_k : \Sigma_A \cap [1] \rightarrow K \cap I_k$  such that there are  $C_1, C_2 > 1$  (only depending on  $k$ ), such that for any  $n \in \mathbb{N}$  and  $A$ -admissible word  $i_1 i_2 \dots i_n$  with  $i_1 = 1$ ,

- (c.1)  $\frac{1}{C_1}\rho^n \leq \text{diam } \pi_k(\Sigma_A \cap [i_1 i_2 \dots i_n]) \leq C_1 \rho^n$ .  
(c.2)  $\frac{1}{C_2}\|M_{i_1} \dots M_{i_n}\| \leq \mu(\pi_k(\Sigma_A \cap [i_1 i_2 \dots i_n])) \leq C_2\|M_{i_1} \dots M_{i_n}\|$ .  
(c.3) *The subintervals  $\text{conv}(\pi_k(\Sigma_A \cap [i_1 i_2 \dots i_n i_{n+1}]))$  (where  $i_{n+1}$ 's are taken so that  $A_{i_n i_{n+1}} = 1$ ) have disjoint interiors. Here  $\text{conv}(Y)$  denotes the convex hull of  $Y$ . Moreover if we set  $[a, b] := \text{conv}(\pi_k(\Sigma_A \cap [i_1 i_2 \dots i_n]))$  and  $[c, d] := \text{conv}(\pi_k(\Sigma_A \cap [i_1 i_2 \dots i_n i_{n+1}]))$ , then the ratios  $\frac{d-c}{b-a}$  and  $\frac{d-a}{b-a}$  only depend on  $i_n$  and  $i_{n+1}$ .*

Most parts of the above proposition were proved or implicit in [12]. However for the convenience of the readers and for completeness, we will provide a detailed constructive proof in the next section.

**Proposition 5.2.** *Let  $m$ ,  $M_i$  ( $1 \leq i \leq m$ ) and  $\{I_k\}_{k \in \Lambda}$  be given as in the above proposition. Let  $(\Sigma, \sigma)$  be the full shift space over the alphabet  $\{1, \dots, m\}$ . For any  $k \in \Lambda$  denote  $\mu_k = \mu|_{I_k}$ , i.e.  $\mu_k$  is the restriction of the measure  $\mu$  on  $I_k$ . Then for any  $q \in \mathbb{R}$ ,*

$$(5.3) \quad \tau(\mu_k, q) = P_M(q) / \log \rho, \quad \forall k \in \Lambda,$$

where  $P_M(q)$  is defined as in (1.5). Moreover  $\tau(\mu_k, q) = \tau(\mu, q)$  for all  $q \geq 0$ .

*Proof.* The proof is identical to that of Proposition 5.7 and Lemma 5.3 in [12].  $\square$

**Proof of Theorem 1.2.** We use the same notion as in Proposition 5.1. Fix one index  $k \in \Lambda$  and consider the measure  $\mu_k$ . For  $t \in \mathbb{R}$ , define

$$E_t = \left\{ y \in \mathbb{R} : \lim_{\delta \rightarrow 0} \frac{\log \mu_k([y - \delta, y + \delta])}{\log \delta} = t \right\}.$$

To prove Theorem 1.2, we shall prove the following two statements respectively:

(S1)  $E_t \neq \emptyset \iff t \in [\alpha_{\min}, \alpha_{\max}]$ , where

$$\alpha_{\min} := \lim_{q \rightarrow +\infty} \frac{\tau(\mu_k, q)}{q} = \lim_{q \rightarrow +\infty} \frac{P_M(q)}{q \log \rho}$$

and

$$\alpha_{\max} := \lim_{q \rightarrow -\infty} \frac{\tau(\mu_k, q)}{q} = \lim_{q \rightarrow -\infty} \frac{P_M(q)}{q \log \rho}$$

(S2) For any  $t \in [\alpha_{\min}, \alpha_{\max}]$ ,

$$\dim_H E_t = \inf_{q \in \mathbb{R}} \{tq - \tau(\mu_k, q)\} = -\frac{1}{\log \rho} \inf_{q \in \mathbb{R}} \{-tq \log \rho + P_M(q)\}.$$

By Lemma 2.3 and Proposition 5.2, we have

$$\lim_{q \rightarrow +\infty} \frac{\tau(\mu_k, q)}{q} = \lim_{q \rightarrow +\infty} \frac{P_M(q)}{q \log \rho} = \frac{\beta_M}{\log \rho}, \quad \lim_{q \rightarrow -\infty} \frac{\tau(\mu_k, q)}{q} = \lim_{q \rightarrow -\infty} \frac{P_M(q)}{q \log \rho} = \frac{\alpha_M}{\log \rho}$$

To prove (S1), we shall present some notation at first. For convenience, we denote  $R(i_1 \dots i_n) := \text{conv}(\pi_k(\Sigma_A \cap [i_1 \dots i_n]))$  for each  $A$ -admissible word  $i_1 \dots i_n$  of length  $n$  with  $i_1 = 1$ , and call it an  $n$ -th *net interval*. By Proposition 5.1(c),  $\frac{1}{C_1}\rho^n \leq \text{diam}(R(i_1 \dots i_n)) \leq C_1\rho^n$  and  $\frac{1}{C_2}\|M_{i_1} \dots M_{i_n}\| \leq \mu_k(R(i_1 \dots i_n)) \leq C_2\|M_{i_1} \dots M_{i_n}\|$ .

Now take any  $y \in \text{supp } \mu_k$ . For any integer  $n > 0$  and  $\delta \in [C_1\rho^n, C_1\rho^{n-1})$ , notice that the interval  $[y - \delta, y + \delta]$  contains at least one  $n$ -th net interval and intersects at most  $\frac{2C_1\rho^{n-1}}{(1/C_1)\rho^n} + 1 = 2(C_1)^2\rho^{-1} + 1$  many different  $n$ -th net intervals. It follows that

$$\frac{1}{C_2} \min_{i_1 \dots i_n} \|M_{i_1} \dots M_{i_n}\| \leq \mu_k([y - \delta, y + \delta]) \leq C_2(2(C_1)^2\rho^{-1} + 1) \max_{i_1 \dots i_n} \|M_{i_1} \dots M_{i_n}\|,$$

where the minimum and the maximum are taken over the set of  $A$ -admissible words of length  $n$ . Applying (2.3) and Lemma 2.3 gives  $\limsup_{\delta \rightarrow 0} \frac{\log \mu_k([y-\delta, y+\delta])}{\log \delta}$ ,  $\liminf_{\delta \rightarrow 0} \frac{\log \mu_k([y-\delta, y+\delta])}{\log \delta} \in [\frac{\beta_M}{\log \rho}, \frac{\alpha_M}{\log \rho}] = [\alpha_{\min}, \alpha_{\max}]$ . Thus we have

$$(5.4) \quad E_t \neq \emptyset \implies t \in [\alpha_{\min}, \alpha_{\max}].$$

This proves the “ $\implies$ ” part of (S1). To prove the other part, take any  $t \in [\alpha_{\min}, \alpha_{\max}]$  and let  $\alpha = t \log \rho$ . Then  $\alpha \in [\alpha_M, \beta_M]$ . Pick an  $A$ -admissible word  $\xi = u_1 u_2 \dots u_s$  with  $u_1 = 1$  such that the net interval  $R(u_1 \dots u_s)$  is strictly contained in the interior of the interval  $R(u_1) = I_k$ . This choice is always possible since  $\mu$  contains no atoms. By Proposition 3.2, there exists a non-empty set  $\Gamma = \Gamma(\xi) \subset E_M(\alpha)$  such that  $\dim_H \Gamma = \dim_H E_M(\alpha)$ , and for any  $x = (x_j)_{j=1}^\infty \in \Gamma$ , there exists a sequence of positive integers  $\{n_p(x)\}_{p=1}^\infty \uparrow \infty$  such that  $n_1(x) = 1$ ,  $\lim_{p \rightarrow \infty} \frac{n_{p+1}(x)}{n_p(x)} = 1$ , and  $\xi$  appears in  $(x_j)_{j=1}^\infty$  at each position  $n_p(x)$ . We shall finish the proof of (S1) by showing that

$$(5.5) \quad \pi_k(\Gamma) \subset E_t.$$

To show (5.5), let  $x = (x_i)_{i=1}^\infty \in \Gamma$ . Then  $x_1 = 1$  and by Proposition 5.1 (b), the words  $x_1 \dots x_n$  are  $A$ -admissible for all  $n$ . Notice that for each  $n$ ,  $\text{diam}(R(x_1 \dots x_n)) \leq C_1 \rho^n$ , hence we have  $[\pi_k x - C_1 \rho^n, \pi_k x + C_1 \rho^n] \supset R(x_1 \dots x_n)$ . It follows that  $\mu_k([\pi_k x - C_1 \rho^n, \pi_k x + C_1 \rho^n]) \geq \mu_k(R(x_1 \dots x_n)) \geq \frac{1}{C_2} \|M_{x_1} \dots M_{x_n}\|$ . Consequently we have

$$(5.6) \quad \begin{aligned} \limsup_{\delta \rightarrow 0} \frac{\log \mu_k([\pi_k x - \delta, \pi_k x + \delta])}{\log \delta} &= \limsup_{n \rightarrow \infty} \frac{\log \mu_k([\pi_k x - C_1 \rho^n, \pi_k x + C_1 \rho^n])}{\log(C_1 \rho^n)} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log(\frac{1}{C_2} \|M_{x_1} \dots M_{x_n}\|)}{\log(C_1 \rho^n)} = \frac{\alpha}{\log \rho} = t. \end{aligned}$$

To see the other direction, notice that the interval  $R(u_1 \dots u_s)$  is contained in the interior of  $R(u_1)$ . Thus there exists  $\gamma > 0$  such that  $[y - \gamma, y + \gamma] \subset R(u_1)$  for each  $y \in R(u_1 \dots u_s)$ . By Proposition 5.1(c.3), for each  $p \in \mathbb{N}$ , the relative position of  $R(x_1 \dots x_{n_p(x)} \dots x_{n_p(x)+s-1})$  in  $R(x_1 \dots x_{n_p(x)})$  is the same as that of  $R(x_{n_p(x)} x_{n_p(x)+1} \dots x_{n_p(x)+s-1})$  in  $R(x_{n_p(x)})$ , i.e., that of  $R(u_1 \dots u_s)$  in  $R(u_1)$ . Therefore for each  $y \in R(x_1 \dots x_{n_p(x)} \dots x_{n_p(x)+s-1})$ ,

$$(5.7) \quad [y - \gamma |R(x_1 \dots x_{n_p(x)-1})|, y + \gamma |R(x_1 \dots x_{n_p(x)-1})|] \subset R(x_1 \dots x_{n_p(x)}).$$

Now for any large  $\ell \in \mathbb{N}$ , there is an integer  $p$  such that

$$\gamma |R(x_1 \dots x_{n_{p+1}(x)-1})| \leq \rho^\ell < \gamma |R(x_1 \dots x_{n_p(x)-1})|. \text{ Therefore by (5.7),}$$

$$\begin{aligned} [\pi_k x - \rho^\ell, \pi_k x + \rho^\ell] &\subset [\pi_k x - \gamma |R(x_1 \dots x_{n_p(x)-1})|, \pi_k x + \gamma |R(x_1 \dots x_{n_p(x)-1})|] \\ &\subset R(x_1 \dots x_{n_p(x)}). \end{aligned}$$

In the meantime,

$$\rho^\ell \geq \gamma |R(x_1 \dots x_{n_{p+1}(x)-1})| \geq \frac{\gamma}{C_1} \rho^{n_{p+1}(x)}.$$

These two inequalities imply

$$\frac{\log \mu_k([\pi_k x - \rho^\ell, \pi_k x + \rho^\ell])}{\log \rho^\ell} \geq \frac{\log \mu_k(R(x_1 \dots x_{n_p(x)}))}{\log(\frac{\gamma}{C_1} \rho^{n_{p+1}(x)})} \geq \frac{\log(C_2 \|M_{x_1} \dots M_{x_{n_p(x)}}\|)}{\log \frac{\gamma}{C_1} + n_{p+1}(x) \log \rho}.$$

Let  $\ell \rightarrow \infty$ . Since  $n_{p+1}(x)/n_p(x)$  tends to 1 as  $p \rightarrow \infty$ , we obtain

$$(5.8) \quad \begin{aligned} \liminf_{\delta \rightarrow 0} \frac{\log \mu_p([\pi_k x - \delta, \pi_k x + \delta])}{\log \delta} &= \liminf_{\ell \rightarrow \infty} \frac{\log \mu_k([\pi_k x - \rho^\ell, \pi_k x + \rho^\ell])}{\ell \log \rho} \\ &\geq \liminf_{p \rightarrow \infty} \frac{\log \|M_{x_1} \dots M_{x_{n_p(x)}}\|}{n_p(x) \log \rho} = t. \end{aligned}$$

By (5.6) and (5.8), we obtain (5.5). This finishes the proof of (S1).

To show (S2), by (5.5) we have  $\dim_H E_t \geq \dim_H \pi_k(\Gamma)$ . Later we shall prove that

$$(5.9) \quad \dim_H \pi_k(F) = \frac{\log m}{-\log \rho} \dim_H F, \quad \forall F \subset \Sigma_A \cap [1].$$

Applying (5.9) we have  $\dim_H E_t \geq \dim_H \pi_k(\Gamma) = \frac{\log m}{-\log \rho} \dim_H \Gamma = \frac{\log m}{-\log \rho} \dim_H E_M(\alpha)$ . Hence using Theorem 1.1, we have

$$\begin{aligned} \dim_H E_t &\geq \frac{1}{-\log \rho} \inf_{q \in \mathbb{R}} \{P_M(q) - \alpha q\} \\ &= \inf_{q \in \mathbb{R}} \left\{ \frac{P_M(q)}{-\log \rho} - \frac{\alpha}{-\log \rho} q \right\} \\ &= \inf_{q \in \mathbb{R}} \{-\tau(\mu_k, q) + tq\}. \end{aligned}$$

However, since  $E_t \neq \emptyset$ , the upper bound  $\dim_H E_t \leq \inf_{q \in \mathbb{R}} \{-\tau(\mu_k, q) + tq\}$  is generic, not depending on the special property of  $\mu_k$  (see, e.g., [7, Theorem 1] or [31, Theorem 4.1]). This finishes the proof of (S2).

Now we prove (5.9). Actually the proof is quite standard. Since  $\frac{1}{C_1} \rho^n \leq \text{diam}(R_{i_1 \dots i_n}) \leq C_1 \rho^n$  for any  $n$ -th net interval  $R_{i_1 \dots i_n}$ , we have

$$|\pi_k x - \pi_k y| \leq C_1 (d(x, y))^{\frac{-\log \rho}{\log m}}, \quad \forall x, y \in \Sigma_A \cap [1].$$

This implies (for a proof, see, e.g., [20, Proposition 2.2].)

$$(5.10) \quad \mathcal{H}^{-s \log m / \log \rho}(\pi_k F) \leq (C_1)^{-s \log m / \log \rho} \mathcal{H}^s(F), \quad \forall s \geq 0, F \subset \Sigma_A \cap [1],$$

where  $\mathcal{H}^u$  denotes the  $u$ -dimensional Hausdorff measure. On the other hand, for each  $s \geq 0$  and  $F \subset \Sigma_A \cap [1]$ , let  $\{\Delta_n\}_{n=1}^\infty$  be a family of intervals such that  $\text{diam}(\Delta_n) \leq \text{diam}(I_k)$ ,  $\Delta_n \cap \pi_k F \neq \emptyset$  and  $\bigcup_{n \geq 1} \Delta_n \supset \pi_k F$ . Then for each  $n$  set

$$G_n = \{A\text{-admissible words } i_1 \dots i_p : i_1 = 1, R_{i_1 \dots i_p} \cap \Delta_n \neq \emptyset, \rho^p \leq \text{diam}(\Delta_n) < \rho^{p-1}\}.$$

By the net property,  $\#G_n \leq \frac{C_1}{\rho} + 1$ , and furthermore the family  $\{\Sigma_A \cap [\eta] : \eta \in \bigcup_{n \geq 1} G_n\}$  is a cover of  $F$  with

$$\sum_{\eta \in \bigcup_{n \geq 1} G_n} \text{diam}([\eta] \cap \Sigma_A)^s \leq \left( \frac{C_1}{\rho} + 1 \right) \sum_{n \geq 1} \text{diam}(\Delta_n)^{-s \log m / \log \rho}.$$

Thus we have

$$\mathcal{H}^{-s \log m / \log \rho}(\pi_k F) \geq (C_1/\rho + 1)^{-1} \mathcal{H}^s(F), \quad \forall s \geq 0, F \subset \Sigma_A \cap [1].$$

Combining it with (5.10) yields (5.9). This finishes the proof of Theorem 1.2.  $\square$

**Remark 5.3.** Applying Theorem 1.2 and Proposition 5.2, we can deduce a global property of  $\mu$ , that is,

$$(5.11) \quad \dim_H E_\mu(\alpha) = \inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\}, \quad \forall \alpha \in [\tau'(+\infty), \tau'(0-)],$$

where  $\tau(q) = \tau(\mu, q)$ . To see this, since  $\tau(q)$  is concave, we have for  $\alpha \in [\tau'(+\infty), \tau'(0-)]$ ,  $\inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\} = \inf_{q \geq 0} \{\alpha q - \tau(q)\}$ . By Proposition 5.2,  $\tau(\mu_k, q) = \tau(q)$  for all  $q \geq 0$  and

$k \in \Lambda$ . Thus by Theorem 1.2, for  $\alpha \in [\tau'(+\infty), \tau'(0-)]$ ,

$$\begin{aligned} \inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\} &= \inf_{q \geq 0} \{\alpha q - \tau(q)\} = \inf_{q \geq 0} \{\alpha q - \tau(\mu_k, q)\} \\ &= \inf_{q \in \mathbb{R}} \{\alpha q - \tau(\mu_k, q)\} = \dim_H E_{\mu_k}(\alpha). \end{aligned}$$

Since  $\dim_H E_\mu(\alpha) \geq \dim_H E_{\mu_k}(\alpha)$  and  $\dim_H E_\mu(\alpha) \leq \inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\}$ , the equality (5.11) follows.

Furthermore if the equality  $\tau(\mu_k, q) = \tau(q)$  holds for all  $q < 0$ , then we have a strong result: *The measure  $\mu$  itself satisfies the complete multifractal formalism.* To see it, we have for all  $\alpha \in [\tau'(+\infty), \tau'(-\infty)]$ ,

$$\inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\} = \inf_{q \in \mathbb{R}} \{\alpha q - \tau(\mu_k, q)\} = \dim_H E_{\mu_k}(\alpha).$$

Since  $\dim_H E_\mu(\alpha) \geq \dim_H E_{\mu_k}(\alpha)$  and  $\dim_H E_\mu(\alpha) \leq \inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\}$ , we have

$$(5.12) \quad \dim_H E_\mu(\alpha) = \inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\}, \quad \forall \alpha \in [\tau'(+\infty), \tau'(-\infty)],$$

On the other hand,  $E_\mu(\alpha) \neq \emptyset$  implies  $\alpha \in [\tau'(+\infty), \tau'(-\infty)]$ . To see it, denote  $a_\epsilon = \sup_{x \in \text{supp}(\mu)} \mu([x - \epsilon, x + \epsilon])$  and  $b_\epsilon = \inf_{x \in \text{supp}(\mu)} \mu([x - \epsilon, x + \epsilon])$ . Denote  $\tau(\epsilon, q) := \sup \sum_i \mu(B_\epsilon(x_i))^q$ , where the supremum is taken over all the families of disjoint balls  $B_\epsilon(x_i)$  of radius  $\epsilon$  and center  $x_i \in \text{supp}(\mu)$ . Then we have

$$\begin{cases} (a_\epsilon)^q \tau(\epsilon, q) \leq \epsilon^{-1} (a_\epsilon)^q, & \text{if } q \geq 0, \\ (b_\epsilon)^q \tau(\epsilon, q) \leq \epsilon^{-1} (b_\epsilon)^q, & \text{if } q < 0. \end{cases}$$

Since by definition  $\tau(q) = \liminf_{\epsilon \rightarrow 0} \log \tau(\epsilon, q) / \log \epsilon$ , we have

$$\begin{cases} q \liminf_{\epsilon \rightarrow 0} \frac{\log a_\epsilon}{\log \epsilon} - 1 \leq \tau(q) \leq q \liminf_{\epsilon \rightarrow 0} \frac{\log a_\epsilon}{\log \epsilon}, & \text{if } q \geq 0, \\ q \limsup_{\epsilon \rightarrow 0} \frac{\log b_\epsilon}{\log \epsilon} - 1 \leq \tau(q) \leq q \limsup_{\epsilon \rightarrow 0} \frac{\log b_\epsilon}{\log \epsilon}. & \text{if } q < 0. \end{cases}$$

This implies

$$\liminf_{\epsilon \rightarrow 0} \frac{\log a_\epsilon}{\log \epsilon} = \lim_{q \rightarrow +\infty} \tau(q)/q = \tau'(+\infty), \quad \limsup_{\epsilon \rightarrow 0} \frac{\log b_\epsilon}{\log \epsilon} = \lim_{q \rightarrow -\infty} \tau(q)/q = \tau'(-\infty).$$

Thus  $E_\mu(\alpha) \neq \emptyset$  implies  $\alpha \in [\tau'(+\infty), \tau'(-\infty)]$ . This fact, together with (5.12), shows that  $\mu$  satisfies the complete multifractal formalism.

## 6. THE STRUCTURE OF SELF-SIMILAR MEASURES SATISFYING THE FINITE TYPE CONDITION

In this section we will outline the structure of self-similar measures satisfying the finite type condition and provide a constructive proof of Proposition 5.1. Especially we shall take the Erdős measure as an example. Almost all the results involved come from [12] and [13].

Let  $S_j x = \rho x + b_j$  ( $j = 1, 2, \dots, \ell$ ) satisfy the finite type condition, i.e.,  $\{S_j\}$  satisfies (5.1). Let  $\mu$  be the self-similar measure generated by  $\{S_j\}_{j=1}^\ell$  with the probability weight  $\{p_j\}_{j=1}^\ell$ . Let  $K$  be the corresponding self-similar set. Without loss of generality, here and afterwards we always assume

$$0 = b_1 < b_2 < \dots < b_\ell = 1 - \rho.$$

Under this assumption, the convex hull of  $K$  is just the interval  $[0, 1]$ .

Write  $\mathcal{A} = \{1, \dots, \ell\}$ . For  $n \in \mathbb{N}$  let  $\mathcal{A}_n$  denote the collection of all indices  $j_1 \dots j_n$  of length  $n$  over  $\mathcal{A}$ . For  $\sigma = j_1 \dots j_n \in \mathcal{A}_n$  write  $S_\sigma = S_{j_1} \circ \dots \circ S_{j_n}$ . We define two families of sets  $P_n^0, P_n^1$  ( $n \geq 0$ ) in the following way:  $P_0^0 = \{0\}$ ,  $P_0^1 = \{1\}$ , and  $P_n^0 = \{S_\sigma(0) : \sigma \in \mathcal{A}_n\}$ ,  $P_n^1 = \{S_\sigma(1) : \sigma \in \mathcal{A}_n\}$

for  $n \geq 1$ . Define  $P_n = P_n^0 \cup P_n^1$  for  $n \geq 0$ . Let  $h_1, \dots, h_{s_n}$  be all the elements of  $P_n$  ranked in the increasing order. Define

$$\mathcal{F}_n = \{[h_j, h_{j+1}] : 1 \leq j < s_n, (h_j, h_{j+1}) \cap K \neq \emptyset\}.$$

Each element in  $\mathcal{F}_n$  is called an  $n$ -th basic net interval. The following facts about basic net intervals can be checked easily: (i)  $\bigcup_{\Delta \in \mathcal{F}_n} \Delta \supset K$  for any  $n \geq 0$ ; (ii) For any  $\Delta_1, \Delta_2 \in \mathcal{F}_n$  with  $\Delta_1 \neq \Delta_2$ ,  $\text{int}(\Delta_1) \cap \text{int}(\Delta_2) = \emptyset$ ; (iii) For any  $\Delta \in \mathcal{F}_n$  ( $n \geq 1$ ), there is a unique element  $\widehat{\Delta} \in \mathcal{F}_{n-1}$  such that  $\widehat{\Delta} \supset \Delta$ .

For each  $\Delta = [a, b] \in \mathcal{F}_n$  ( $n \geq 0$ ), we will define a positive number  $\ell_n(\Delta)$ , a vector  $V_n(\Delta)$  and a positive integer  $r_n(\Delta)$ . If  $\Delta = [0, 1] \in \mathcal{F}_0$ , we define  $\ell_0(\Delta) = 1$ ,  $V_0(\Delta) = 0$  and  $r_0(\Delta) = 1$ . Otherwise for  $n \geq 1$ , we define  $\ell_n(\Delta)$  and  $V_n(\Delta)$  directly by

$$\ell_n(\Delta) = \rho^{-n}(b - a)$$

and

$$V_n(\Delta) = (a_1, \dots, a_k).$$

where  $a_1, \dots, a_k$  (ranked in the increasing order) are all the element of the following set

$$\{\rho^{-n}(a - S_\sigma(0)) : \sigma \in \mathcal{A}_n, S_\sigma(K) \cap (a, b) \neq \emptyset\}.$$

Denote by  $v_n(\Delta)$  the dimension of  $V_n(\Delta)$ , that is,  $v_n(\Delta) = k$ . We define  $r_n(\Delta)$  in the following way: let  $\widehat{\Delta}$  be the unique interval in  $\mathcal{F}_{n-1}$  containing  $\Delta$ , and  $\Delta_1, \dots, \Delta_k$  (ranked in the increasing order) be all the elements in  $\mathcal{F}_n$  satisfying  $\Delta_j \subset \widehat{\Delta}$ ,  $\ell_n(\Delta_j) = \ell_n(\Delta)$ ,  $V_n(\Delta_j) = V_n(\Delta)$  for  $1 \leq j \leq k$ . Define  $r_n(\Delta)$  to be the integer  $r$  so that  $\Delta_r = \Delta$ . For convenience, we call the triple

$$\mathcal{C}_n(\Delta) := (\ell_n(\Delta); V_n(\Delta); r_n(\Delta))$$

the  $n$ -th characteristic vector of  $\Delta$ , or simply characteristic vector of  $\Delta$ . The vector  $\mathcal{C}_n(\Delta)$  contains the information about the length and neighborhood relation of  $\Delta$ . Define

$$(6.1) \quad \Omega = \{\mathcal{C}_n(\Delta) : n \geq 0, \Delta \in \mathcal{F}_n\}.$$

For any  $\alpha \in \Omega$ , we write for simplicity

$$(6.2) \quad \ell(\alpha) = \ell_n(\Delta), V(\alpha) = V_n(\Delta), v(\alpha) = v_n(\Delta), r(\alpha) = r_n(\Delta),$$

if  $\Delta \in \mathcal{F}_n$  and  $\mathcal{C}_n(\Delta) = \alpha$ .

**Lemma 6.1.** (i) For a given  $\Delta \in \mathcal{F}_n$  ( $n \geq 0$ ), let  $\Delta_1, \dots, \Delta_k$  (ranked in the increasing order) be all the elements in  $\mathcal{F}_{n+1}$  which are subintervals of  $\Delta$ . Then the number  $k$ , the vectors  $\mathcal{C}_{n+1}(\Delta_i)$  ( $1 \leq i \leq k$ ) are determined by  $\ell_n(\Delta)$  and  $V_n(\Delta)$  (thus they are determined by  $\mathcal{C}_n(\Delta)$ ).

(ii) The set  $\Omega$  is finite.

**Proof.** See [12, Lemma 2.1 and Lemma 2.3]. □

Let  $\Omega^*$  denote the collection of all finite words over  $\Omega$ . For any  $\alpha \in \Omega$ , pick  $n$  and  $\Delta \in \mathcal{F}_n$  such that  $\alpha = \mathcal{C}_n(\Delta)$ . Let  $\Delta_1, \dots, \Delta_k$  (ranked in the increasing order) be all the elements in  $\mathcal{F}_{n+1}$  which are subintervals of  $\Delta$ . Write  $\alpha_j = \mathcal{C}_{n+1}(\Delta_j)$  for  $1 \leq j \leq k$ . By Lemma 6.1, the word  $\alpha_1 \dots \alpha_k$  depend only on  $\alpha$  (independent of the choice of  $n$  and  $\Delta$ ). We define  $\zeta : \Omega \rightarrow \Omega^*$  by

$$\zeta(\alpha) = \alpha_1 \dots \alpha_k.$$

Define a 0-1 matrix  $A$  on  $\Omega \times \Omega$  in the following way:

$$(6.3) \quad A_{\alpha, \beta} = \begin{cases} 1 & \text{if } \beta \text{ is a letter of } \zeta(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

A word  $\beta_1 \dots \beta_n \in \Omega^*$  is called *admissible* if  $A_{\beta_j, \beta_{j+1}} = 1$  for  $1 \leq j < n$ .

Now we would like to associate each basic net interval with a unique admissible word. For each  $\Delta \in \mathcal{F}_n$  ( $n \geq 0$ ), we list the intervals

$$\Delta^0, \Delta^1, \dots, \Delta^n$$

such that  $\Delta^n = \Delta$ , and  $\Delta^j$  ( $j = 0, \dots, n-1$ ) is the unique element in  $\mathcal{F}_j$  such that  $\Delta^j \supset \Delta^{j+1}$ . The word  $\alpha_0 \alpha_1 \dots \alpha_n$ , where  $\alpha_i = \mathcal{C}_i(\Delta^i)$  for  $0 \leq i \leq n$ , is called the *symbolic expression* for  $\Delta$ . The introduction of the third term in a characteristic vector guarantees that two different net intervals have different symbolic expressions. Each admissible word of length  $n+1$  starting from  $\gamma_0 := \mathcal{C}_0([0, 1])$  is a symbolic expression of some  $n$ -th basic net interval.

Now we analyze the distribution of  $\mu$  on basic net intervals. Let  $\Delta = [a, b]$  be an  $n$ -th basic net interval. Write  $V_n(\Delta) = (a_1, \dots, a_{v_n(\Delta)})$ . Iterating (5.2)  $n$  times we obtain

$$\mu(\Delta) = \sum_{\sigma \in \mathcal{A}_n} p_\sigma \mu(S_\sigma^{-1}(\Delta)),$$

where  $p_\sigma$  denotes the product  $p_{j_1} \dots p_{j_n}$  for  $\sigma = j_1 \dots j_n$ . Since  $\mu$  is a non-atomic measure supported on  $K$ , we have

$$\begin{aligned} \mu(\Delta) &= \sum_{\sigma \in \mathcal{A}_n: S_\sigma(K) \cap (a, b) \neq \emptyset} p_\sigma \mu(S_\sigma^{-1}(\Delta)) \\ &= \sum_{i=1}^{v_n(\Delta)} \sum_{\sigma \in \mathcal{A}_n: \rho^{-n}(a - S_\sigma(0)) = a_i} p_\sigma \mu(S_\sigma^{-1}(\Delta)) \\ (6.4) \quad &= \sum_{i=1}^{v_n(\Delta)} \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{\sigma \in \mathcal{A}_n: S_\sigma(0) = a - \rho^n a_i} p_\sigma. \end{aligned}$$

Define a  $v_n(\Delta)$ -dimensional row vector  $Q_n(\Delta) = (q_1, \dots, q_{v_n(\Delta)})$  by

$$(6.5) \quad q_i = \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{\sigma \in \mathcal{A}_n: S_\sigma(0) = a - \rho^n a_i} p_\sigma, \quad i = 1, \dots, v_n(\Delta).$$

We call  $Q_n(\Delta)$  the *vector form* of  $\mu$  on  $\Delta$ . By (6.4),  $\mu(\Delta) = \|Q_n(\Delta)\| := \sum_{i=1}^{v_n(\Delta)} q_i$ . Moreover,  $Q_n(\Delta)$  is always a positive  $v_n(\Delta)$ -dimensional vector for any  $n \geq 0$  and  $\Delta \in \mathcal{F}_n$ .

For any  $\Delta \in \mathcal{F}_n$  ( $n \geq 1$ ), denote by  $\widehat{\Delta}$  the unique element in  $\mathcal{F}_{n-1}$  so that  $\widehat{\Delta} \supset \Delta$ . Assume  $\Delta = [a, b]$  and  $\widehat{\Delta} = [c, d]$ . Write  $V_n(\Delta) = (a_1, \dots, a_{v_n(\Delta)})$  and  $V_{n-1}(\widehat{\Delta}) = (c_1, \dots, c_{v_{n-1}(\widehat{\Delta})})$ . Define for any  $i \in \{1, \dots, v_n(\Delta)\}$  and  $j \in \{1, \dots, v_{n-1}(\widehat{\Delta})\}$ ,

$$w_{j,i} = \begin{cases} p_s & \exists s \in \mathcal{A} \text{ so that } c - \rho^{n-1} c_j + \rho^{n-1} b_s = a - \rho^n a_i, \\ 0 & \text{otherwise.} \end{cases}$$

Define a  $v_{n-1}(\widehat{\Delta}) \times v_n(\Delta)$  matrix  $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)) = (t_{j,i})$  by

$$t_{j,i} = \frac{w_{j,i} \mu([a_i, a_i + \ell_n(\Delta)])}{\mu([c_j, c_j + \ell_{n-1}(\widehat{\Delta})])}, \quad 1 \leq j \leq v_{n-1}(\widehat{\Delta}), \quad 1 \leq i \leq v_n(\Delta).$$

We have

$$Q_n(\Delta) = Q_{n-1}(\widehat{\Delta}) T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)).$$

Since  $\rho^{-n}(c - a)$  depends only on  $\mathcal{C}_{n-1}(\widehat{\Delta})$  and  $\mathcal{C}_n(\Delta)$ , so does  $(w_{j,i})$ . Thus  $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta))$  depends only on  $\mathcal{C}_{n-1}(\widehat{\Delta})$  and  $\mathcal{C}_n(\Delta)$ . Hence we have

**Proposition 6.2.** ([12, Theorem 3.3]) *Let  $T(\alpha, \beta)$  ( $\alpha, \beta \in \Omega$ ,  $A_{\alpha, \beta} = 1$ ) be a family of non-negative matrices defined as above, then for any  $\Delta \in \mathcal{F}_n$ ,*

$$Q_n(\Delta) = T(\gamma_0, \gamma_1) \dots T(\gamma_{n-1}, \gamma_n),$$

where  $\gamma_0 \dots \gamma_n$  is the symbolic expression of  $\Delta$ . □

Moreover we can express precisely the entries of the product  $T(\alpha_1, \alpha_2) \dots T(\alpha_{n-1}, \alpha_n)$  for a given admissible word  $\alpha_1 \dots \alpha_n$ . To see this, choose  $t \in \mathbb{N}$  and  $\Delta = [a, b] \in \mathcal{F}_t$  so that  $\mathcal{C}_t(\Delta) = \alpha_1$ . Assume that the symbolic expression of  $\Delta$  is  $\gamma_0 \dots \gamma_{t-1} \alpha_1$ . Then there is a unique  $\Delta' = [e, f] \in \mathcal{F}_{t+n-1}$  whose symbolic expression is  $\gamma_0 \dots \gamma_{t-1} \alpha_1 \dots \alpha_n$ . Write  $V_t(\Delta) = (a_1, \dots, a_{v_t(\Delta)})$  and  $V_{t+n-1}(\Delta') = (e_1, \dots, e_{v_{t+n-1}(\Delta')})$ . Denote for simplicity  $X = T(\alpha_1, \alpha_2) \dots T(\alpha_{n-1}, \alpha_n)$ . Then from the construction of  $T(\alpha, \beta)$ , we have by induction that

**Proposition 6.3.** ([12, Proposition 3.5]) *For any  $1 \leq j \leq v_t(\Delta)$ , and  $1 \leq i \leq v_{t+n-1}(\Delta')$ ,*

$$X_{j,i} = \frac{\mu([e_i, e_i + \ell_{t+n-1}(\Delta')])}{\mu([a_j, a_j + \ell_t(\Delta)])} \sum_{\xi \in \mathcal{A}_{n-1}: a - \rho^t a_j + \rho^t S_\xi(0) = e - \rho^{t+n-1} e_i} p_\xi.$$

A non-empty subset  $\widehat{\Omega}$  of  $\Omega$  is said to be an *essential class* of  $\Omega$  if it satisfies: (i)  $\{\beta \in \Omega : A_{\alpha, \beta} = 1\} \subset \widehat{\Omega}$  for any  $\alpha \in \widehat{\Omega}$ ; (ii) for any  $\alpha, \beta \in \widehat{\Omega}$ , there exist  $\gamma_1, \dots, \gamma_n \in \widehat{\Omega}$  such that  $\gamma_1 = \alpha$ ,  $\gamma_n = \beta$  and  $A_{\gamma_i, \gamma_{i+1}} = 1$  for  $1 \leq i \leq n-1$ . The existence of at least one essential class is well known (see, e.g., Lemma 1.1 of [45]). In the following we prove

**Lemma 6.4.** (i)  $\Omega$  has exactly one essential class  $\widehat{\Omega}$ .

- (ii) For any  $\alpha \in \Omega$ ,  $k \in \{1, \dots, v(\alpha)\}$  and  $\eta \in \widehat{\Omega}$ , there exists an admissible word  $\alpha_1 \dots \alpha_j$  with  $j \geq 2$  such that  $\alpha_1 = \alpha$ ,  $\alpha_j = \eta$  and all the entries of the  $k$ -th row of the matrix  $T(\alpha_1, \alpha_2) \dots T(\alpha_{j-1}, \alpha_j)$  are positive.
- (iii) There exist two constants  $N \in \mathbb{N}$  and  $C > 0$  such that for any  $\Delta \in \mathcal{F}_n$  and any  $\eta \in \widehat{\Omega}$ , there exist  $s \leq N$  and  $\widehat{\Delta} \subset \Delta$  with  $\widehat{\Delta} \in \mathcal{F}_{n+s}$  and  $\mathcal{C}_{n+s}(\widehat{\Delta}) = \eta$  such that  $\mu(\widehat{\Delta}) \geq C\mu(\Delta)$ .

**Proof.** We adopt a similar method used in the proof of Lemma 4.4 in [12]. To show that  $\Omega$  has exactly one essential class, it suffices to prove that there exists an element  $\gamma$  in  $\Omega$  such that for any  $\alpha \in \Omega$ , there is an admissible word beginning with  $\alpha$  and ending by  $\gamma$ . Equivalently, one needs to show that for any  $n$  and  $\Delta_1 \in \mathcal{F}_n$ , there exists  $\Delta_2 \subset \Delta_1$  with  $\Delta_2 \in \mathcal{F}_{n'}$  for some  $n' > n$  and  $\mathcal{C}_{n'}(\Delta_2) = \gamma$ .

To achieve this, we pick  $\beta \in \Omega$  such that (i)  $\ell(\beta) = \min\{\ell(\eta) : \eta \in \Omega\}$ ; (ii)  $v(\beta) = \max\{v(\eta) : \eta \in \Omega, \ell(\eta) = \ell(\beta)\}$ , where  $v(\cdot)$  and  $\ell(\cdot)$  are defined as in (6.2). Fix an element  $\gamma \in \Omega$  such that  $A_{\beta, \gamma} = 1$ . Choose  $q \in \mathbb{N}$  and  $\Delta = [c, d] \in \mathcal{F}_q$  such that  $\mathcal{C}_q(\Delta) = \beta$ . Write  $V_q(\Delta) = (c_1, \dots, c_{v_q(\Delta)})$ .

Choose arbitrarily  $n \in \mathbb{N}$  and  $\Delta_1 = [a, b] \in \mathcal{F}_n$ . Write  $V_n(\Delta_1) = (a_1, \dots, a_{v_n(\Delta_1)})$ . Fix  $k \in \{1, \dots, v_n(\Delta_1)\}$ . By the definition of  $V_n(\Delta_1)$ , there exists  $\sigma \in \mathcal{A}_n$  with  $S_\sigma(0) = a - \rho^n a_k$  and  $S_\sigma(K) \cap (a, b) \neq \emptyset$ . Find a large integer  $l$  and  $\phi \in \mathcal{A}_l$  so that  $S_{\sigma\phi}(K) \subset (a, b)$  and thus  $S_{\sigma\phi}([0, 1]) \subset (a, b)$ , where  $\sigma\phi$  denotes the concatenation of  $\sigma$  and  $\phi$ .

Denote  $\widehat{\Delta} = S_{\sigma\phi}(\Delta)$ . It is clear  $\widehat{\Delta} \subset (a, b)$  since  $S_{\sigma\phi}([0, 1]) \subset (a, b)$ . We claim that  $\widehat{\Delta} \in \mathcal{F}_{n+l+q}$  with  $V_{n+l+q}(\widehat{\Delta}) = V_q(\Delta)$  and  $\ell_{n+l+q}(\widehat{\Delta}) = \ell_q(\Delta)$ . First we show  $\widehat{\Delta} \in \mathcal{F}_{n+l+q}$  and  $\ell_{n+l+q}(\widehat{\Delta}) = \ell_q(\Delta)$ . To see this, we observe that the two endpoints of  $\widehat{\Delta}$  belong to the set  $P_{n+l+q}$  since those of  $\Delta$  belong to  $P_q$ . In the meantime  $\text{int}(\widehat{\Delta}) \cap K \neq \emptyset$  since  $\text{int}(\Delta) \cap K \neq \emptyset$ . Therefore,  $\widehat{\Delta}$  contains at least one element in  $\mathcal{F}_{n+l+q}$ . On the other hand the minimality of  $\ell(\beta)$  shows that each  $(n+l+q)$ -th

basic net interval has length at least  $\rho^{n+l+q}\ell(\beta)$ , i.e., the length of  $\widehat{\Delta}$ . Combining these two facts we have  $\widehat{\Delta} \in \mathcal{F}_{n+l+q}$  and  $\ell_{n+l+q}(\widehat{\Delta}) = \ell(\beta) = \ell_q(\Delta)$ . To show  $V_{n+l+q}(\widehat{\Delta}) = V_q(\Delta)$ , by the maximum of  $v(\beta)$  it suffices to show each coordinate of the vector  $V_q(\Delta)$  is a coordinate of  $V_{n+l+q}(\widehat{\Delta})$ . To prove this, note that for any  $1 \leq u \leq v_q(\Delta)$ , there exists  $\psi \in \mathcal{A}_q$  such that  $S_\psi(0) = c - \rho^q c_u$  and  $S_\psi(K) \cap (c, d) \neq \emptyset$ . Therefore,  $S_{\sigma\phi\psi}(0) = S_{\sigma\phi}(c) - \rho^{n+l+q}c_u$  and  $S_{\sigma\phi\psi}(K) \cap \text{int}(S_{\sigma\phi}(\Delta)) \neq \emptyset$ . Note that  $\widehat{\Delta} = S_{\sigma\phi}(\Delta)$  and  $S_{\sigma\phi}(c)$  is the left endpoint of  $\widehat{\Delta}$ . By the definition of  $V_{n+l+q}(\widehat{\Delta})$ ,  $c_u$  is a coordinate of  $V_{n+l+q}(\widehat{\Delta})$ . Thus the claim follows. Since the first two terms of  $\mathcal{C}_{n+l+q}(\widehat{\Delta})$  are the same as  $\mathcal{C}_q(\Delta) = \beta$  and  $A_{\beta,\gamma} = 1$ , by Lemma 6.1, there exists an  $(n+l+q+1)$ -th basic net interval  $\Delta_2 \subset \widehat{\Delta}$  such that  $\mathcal{C}_{n+l+q+1}(\Delta_2) = \gamma$ . This finishes the proof of part (i) of the lemma.

To show part (ii), we use the same notation as in the above proof. Denote by  $\gamma_0 \dots \gamma_{n-1}\alpha$  the symbolic expression of  $\Delta_1$ . Since  $\widehat{\Delta} \in \mathcal{F}_{n+l+q}$  and  $\widehat{\Delta} \subset \Delta_1$ , we can denote by  $\gamma_0 \dots \gamma_{n-1}\alpha\gamma_{n+1} \dots \gamma_{n+l+q}$  the symbolic expression of  $\widehat{\Delta}$ . Especially as proved above,  $\gamma_{n+l+q}$  satisfies  $V(\gamma_{n+l+q}) = V(\beta)$  and  $\ell(\gamma_{n+l+q}) = \ell(\beta)$ . Denote

$$X = T(\alpha, \gamma_{n+1})T(\gamma_{n+1}, \gamma_{n+2}) \dots T(\gamma_{n+l+q-1}, \gamma_{n+l+q}).$$

By Proposition 6.3, for any  $1 \leq u \leq v(\beta) = v(\gamma_{n+l+q})$ ,

$$(6.6) \quad X_{k,u} = \frac{\mu([c_u, c_u + \ell(\beta)])}{\mu([a_k, a_k + \ell(\alpha)])} \cdot \sum_{\xi \in \mathcal{A}_{l+q}: a - \rho^n a_k + \rho^n S_\xi(0) = S_{\sigma\phi}(c) - \rho^{n+l+q}c_u} p_\xi.$$

Recall that we have proved in last paragraph that for each  $1 \leq u \leq v(\beta)$ , there exists  $\psi \in \mathcal{A}_q$  such that  $S_{\sigma\phi\psi}(0) = S_{\sigma\phi}(c) - \rho^{n+l+q}c_u$ . Note that

$$S_{\sigma\phi\psi}(0) = S_\sigma(0) + \rho^n S_{\phi\psi}(0) = a - \rho^n a_k + \rho^n S_{\phi\psi}(0).$$

By (6.6),  $X_{k,u} > 0$ . Therefore

$$(6.7) \quad \mathbf{e}_k T(\alpha, \gamma_{n+1})T(\gamma_{n+1}, \gamma_{n+2}) \dots T(\gamma_{n+l+q-1}, \gamma_{n+l+q}) > \mathbf{0},$$

where  $\mathbf{e}_k$  denotes the  $v(\alpha)$ -dimensional row vector whose  $k$ -th coordinate is 1 and all other coordinates are 0.

Choose an admissible sequence  $\eta_1 \dots \eta_t$  such that  $\eta_1 = \gamma$  and  $\eta_t = \eta$ . By (6.7),

$$(6.8) \quad \mathbf{e}_k T(\alpha, \gamma_{n+1}) \dots T(\gamma_{n+l+q-1}, \gamma_{n+l+q})T(\gamma_{n+l+q}, \eta_1)T(\eta_1, \eta_2) \dots T(\eta_{t-1}, \eta_t) > \mathbf{0},$$

That is, all the entries of the  $k$ -th row of the matrix

$$T(\alpha, \gamma_{n+1})T(\gamma_{n+1}, \gamma_{n+2}) \dots T(\gamma_{n+l+q-1}, \gamma_{n+l+q})T(\gamma_{n+l+q}, \eta_1)T(\eta_1, \eta_2) \dots T(\eta_{t-1}, \eta_t)$$

are positive, which completes the proof of part (ii).

Part (iii) follows from part (ii). To see it, by part (ii) we can construct a finite family of admissible words  $\{\alpha_1 \dots \alpha_j\}_{\alpha, k, \eta}$  when  $\alpha, k$  and  $\eta$  are taken over the set  $\Omega$ ,  $\{v(1), \dots, v(\alpha)\}$  and  $\widehat{\Omega}$  respectively. Let  $N$  denote the largest length of the words in this family. Let  $D$  denote the smallest non-zero entry appear in the constructed matrices  $T(\alpha_1, \alpha_2) \dots T(\alpha_{j-1}, \alpha_j)$ .

Now fix  $\Delta \in \mathcal{F}_n$  and  $\eta \in \widehat{\Omega}$ . Assume  $\mathcal{C}_n(\Delta) = \alpha$ . Choose  $k \in \{1, \dots, v(\alpha)\}$  such that  $k$ -th entry of  $Q_n(\Delta)$  is larger than or equals to  $\frac{1}{v(\alpha)}\|Q_n(\Delta)\| = \frac{1}{v(\alpha)}\mu(\Delta)$ . Let  $\alpha_1 \dots \alpha_j$  be a constructed admissible word beginning with  $\alpha$  and ending by  $\eta$ , such that the  $k$ -th row of the matrix  $T(\alpha_1, \alpha_2) \dots T(\alpha_{j-1}, \alpha_j)$  are positive. Let  $W$  be the symbolic expression of  $\Delta$ . Then there exists a unique one  $\widehat{\Delta} \subset \Delta$  with  $\widehat{\Delta} \in \mathcal{F}_{n+j-1}$  such that symbolic expression of  $\widehat{\Delta}$  is  $W\alpha_2 \dots \alpha_j$ . Then

$$\mu(\widehat{\Delta}) = \|Q_n(\Delta)T(\alpha_1, \alpha_2) \dots T(\alpha_{j-1}, \alpha_j)\| \geq \frac{1}{Dv(\alpha)}\mu(\Delta).$$

This finishes the proof by letting  $C = \inf_{\alpha \in \Omega} \frac{1}{Dv(\alpha)}$ .  $\square$

Let  $\widehat{\Omega}$  be the essential class of  $\Omega$ . In what follows we always denote  $m := \#\widehat{\Omega}$  and write  $\widehat{\Omega} = \{\eta_1, \dots, \eta_m\}$ . Set  $d = \sum_{i=1}^m v(\eta_i)$ , where  $v(\cdot)$  is defined as in (6.2). In the following we construct a family of  $d \times d$  matrices  $\{M_i\}_{i=1}^m$ . For any  $1 \leq i \leq m$ , define  $M_i$  to be the partitioned matrix

$$M_i = \begin{bmatrix} U_{1,1}^i & U_{1,2}^i & \cdots & U_{1,m}^i \\ U_{2,1}^i & U_{2,2}^i & \cdots & U_{2,m}^i \\ \vdots & \vdots & \ddots & \vdots \\ U_{m,1}^i & U_{m,2}^i & \cdots & U_{m,m}^i \end{bmatrix},$$

where for each  $1 \leq j, k \leq m$ ,  $U_{j,k}^i$  is a  $v(\eta_j) \times v(\eta_k)$  matrix defined by

$$U_{j,k}^i = \begin{cases} T(\eta_j, \eta_i) & \text{if } k = i \text{ and } A_{\eta_j, \eta_i} = 1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Let  $J_0 \in \mathcal{F}_{n_0}$  so that  $\mathcal{C}_{n_0}(J_0) = \eta_1$ . Denote  $\Theta_0 \in \Omega^*$  by  $\Theta_0 = \gamma_0 \dots \gamma_{n_0-1} \eta_1$  the symbolic expression of  $J_0$ . Given  $\Delta \in \mathcal{F}_n$  ( $n \geq n_0$ ) with  $\Delta \subset J_0$ , define  $\widehat{Q}_n(\Delta)$  to be the partitioned vector  $(W_1, \dots, W_m)$ , where  $W_i$  is a  $v(\eta_i)$ -dimensional row vector defined by

$$W_i = \begin{cases} Q_n(\Delta) & \text{if } \eta_i = \mathcal{C}_n(\Delta), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

It is clear that  $\widehat{Q}_n(\Delta)$  is a  $d$ -dimensional row vector, which is called the *uniform vector form* of  $\mu$  on  $\Delta$ . We have

**Lemma 6.5.** (i) *The matrix  $H := \sum_{i=1}^m M_i$  is irreducible. That is, there exists an integer  $r > 0$  such that  $H^r > \mathbf{0}$ .*

(ii) *Given  $\Delta \in \mathcal{F}_{n_0+n}$  ( $n \geq 1$ ) with  $\Delta \subset J_0$ , we have*

$$\widehat{Q}_{n_0+n}(\Delta) = \widehat{Q}_{n_0}(J_0) M_{i_1} \dots M_{i_n} \approx \mathbf{e} M_1 M_{i_1} \dots M_{i_n},$$

where  $\Theta_0 \eta_{i_1} \dots \eta_{i_n}$  is the symbolic expression of  $\Delta$ ,  $\mathbf{e}$  denotes the  $d$ -dimensional row vector of which each entry equals 1, and the constant involved in “ $\approx$ ” only depends on  $J_0$ . Hence  $\mu(\Delta) \approx \|M_1 M_{i_1} \dots M_{i_n}\|$ .

(iii)  *$M_{i_1} \dots M_{i_k} \neq \mathbf{0}$  if and only if  $\eta_{i_1} \dots \eta_{i_k}$  is an admissible sequence.*

**Proof.** See [12, Lemma 4.1 and Proposition 4.2].  $\square$

**Proof of Proposition 5.1.** Let  $m$ ,  $d$  and  $M_i$  ( $1 \leq i \leq m$ ) be constructed as above. Without confusion we define an  $m \times m$  0-1 matrix  $A = (A_{i,j})_{1 \leq i, j \leq m}$  by

$$A_{i,j} = A_{\eta_i, \eta_j},$$

where  $A_{\alpha, \beta}$  ( $\alpha, \beta \in \Omega$ ) is defined as in (6.3). Since  $\widehat{\Omega}$  is an essential class of  $\Omega$ , the matrix  $(A_{i,j})_{1 \leq i, j \leq m}$  is irreducible. Then part (a) and (b) of Proposition 5.1 follows from Lemma 6.5 (i) and (ii). Define  $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$  and  $\Sigma_A = \{(x_i)_{i=1}^{\infty} \in \Sigma : A_{x_i, x_{i+1}} = 1 \text{ for } i \geq 1\}$ . Moreover we define  $\Lambda \subset \Omega^*$  by

$$\Lambda = \{A\text{-admissible word } \alpha_0 \alpha_1 \dots \alpha_n : \alpha_0 = \gamma_0, \alpha_n = \eta_1, \alpha_i \neq \eta_1 \text{ for } 1 \leq i < n\}.$$

For any  $W = \alpha_0 \alpha_1 \dots \alpha_n \in \Lambda$ , we denote by  $I_W$  the  $n$ -th basic net interval whose symbolic expression is  $W = \alpha_0 \alpha_1 \dots \alpha_n$ . By the definition of  $\Lambda$  and the structure of basic net intervals,  $\text{int}(I_W) \cap \text{int}(I_{W'}) = \emptyset$  for different words  $W, W' \in \Lambda$ .

Now we show  $\mu(\bigcup_{W \in \Lambda} I_W) = 1$  by contradiction. Assume that  $\mu(\bigcup_{W \in \Lambda} I_W) = t < 1$ . Then for any  $\epsilon > 0$ , there exists  $n \in \mathbb{N}$  such that

$$t - \epsilon \leq \sum_{W \in \Lambda, |W| \leq n} \mu(I_W) \leq t,$$

where  $|W|$  denotes the length of the word  $W$ . Let  $\mathcal{F}'_n$  denotes the collection of all  $n$ -th basic net intervals that have no intersection with the interior of  $\bigcup_{W \in \Lambda, |W| \leq n} I_W$ . Then  $\bigcup_{\Delta \in \mathcal{F}'_n} \Delta \cup (\bigcup_{W \in \Lambda, |W| \leq n} I_W) \supset K$ , thus  $\sum_{\Delta \in \mathcal{F}'_n} \mu(\Delta) \geq 1 - t$ . By Lemma 6.4 (iii), for any  $\Delta \in \mathcal{F}'_n$  there exists a corresponding  $U \in \Lambda$  with  $|U| > n$  such that  $I_U \subset \Delta$  and  $\mu(I_U) \geq C\mu(\Delta)$  for a constant  $C > 0$  independent of  $n$ . Therefore

$$\sum_{U \in \Lambda, |U| > n} \mu(I_U) \geq C \sum_{\Delta \in \mathcal{F}'_n} \mu(\Delta) \geq C(1 - t),$$

which leads to a contradiction with the fact  $\sum_{U \in \Lambda, |U| > n} \mu(I_U) \leq \epsilon$ , since  $\epsilon$  can be taken arbitrarily small.

Now for  $W = \alpha_0 \alpha_1 \dots \alpha_n \in \Lambda$  we define the map  $\pi_W : \Sigma_A \cap [1] \rightarrow K \cap I_W$  by

$$\pi_W(x) = \bigcap_{k=2}^{\infty} \Delta_{x_1 x_2 \dots x_k}^W, \quad \forall x = (x_i)_{i=1}^{\infty},$$

where  $\Delta_{x_1 x_2 \dots x_k}^W$  denotes the basic net interval whose symbolic expression is  $W \eta_{x_2} \dots \eta_{x_k}$ . By the structure of basic net intervals,  $\text{conv}(\pi_W(\Sigma_A \cap [x_1 x_2 \dots x_k])) = \Delta_{x_1 x_2 \dots x_k}^W$ . From this fact and Lemma 6.5 (ii), part (c) of Proposition 5.1 follows. This finishes the proof.  $\square$

**Example 6.6.** Let  $\rho = \frac{\sqrt{5}-1}{2}$  and let  $\mu$  be the self-similar measure generated by the family of maps  $\{S_1 x = \rho x, S_2 x = \rho x + 1 - \rho\}$  and the probability weight  $\{1/2, 1/2\}$ . The measure  $\mu$ , which is also called the Erdős measure, is the well known Bernoulli convolution associated with the golden ratio. The structure of this measure has been extensively studied in [13]. For this measure we have  $\Omega = \{\alpha_1, \alpha_2, \dots, \alpha_7\}$ , where

$$\begin{aligned} \alpha_1 &= (1; 0; 1), & \alpha_2 &= (\rho; 0; 1), & \alpha_3 &= (1 - \rho; (0, \rho); 1), & \alpha_4 &= (\rho; 1 - \rho; 1), \\ \alpha_5 &= (\rho; (0, 1 - \rho); 1), & \alpha_6 &= (2\rho - 1; 1 - \rho; 1), & \alpha_7 &= (1 - \rho; (0, \rho); 2). \end{aligned}$$

The map  $\zeta : \Omega \rightarrow \Omega^*$  is given by

$$\begin{aligned} \zeta(\alpha_1) &= \alpha_2 \alpha_3 \alpha_4, & \zeta(\alpha_2) &= \alpha_2 \alpha_3, & \zeta(\alpha_3) &= \alpha_5, & \zeta(\alpha_4) &= \alpha_3 \alpha_4, \\ \zeta(\alpha_5) &= \alpha_3 \alpha_6 \alpha_7, & \zeta(\alpha_6) &= \alpha_3, & \zeta(\alpha_7) &= \alpha_5. \end{aligned}$$

Let  $A$  be defined as (6.3). The matrices  $\{T(\alpha, \beta) : \alpha, \beta \in \Omega, A_{\alpha, \beta} = 1\}$  are given by

$$\left\{ \begin{array}{l} T(\alpha_1, \alpha_2) = 1/3, & T(\alpha_1, \alpha_3) = [1/6, 1/6], & T(\alpha_1, \alpha_4) = 1/3, \\ T(\alpha_2, \alpha_2) = 1/2, & T(\alpha_2, \alpha_3) = [1/4, 1/4], \\ T(\alpha_3, \alpha_5) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ T(\alpha_4, \alpha_3) = [1/4, 1/4], & T(\alpha_4, \alpha_4) = 1/2, \\ T(\alpha_5, \alpha_3) = \begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{bmatrix}, & T(\alpha_5, \alpha_6) = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}, & T(\alpha_5, \alpha_7) = \begin{bmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{bmatrix}, \\ T(\alpha_6, \alpha_3) = [1/2, 1/2], \\ T(\alpha_7, \alpha_5) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{array} \right.$$

The essential class of  $\Omega$  is  $\widehat{\Omega} = \{\alpha_3, \alpha_5, \alpha_6, \alpha_7\}$ . For convenience we relabel the elements of  $\widehat{\Omega}$  by

$$1 := \alpha_3, \quad 2 := \alpha_5, \quad 3 := \alpha_6, \quad 4 := \alpha_7.$$

The matrices  $M_i$  ( $1 \leq i \leq 4$ ) are given by

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For this special case the  $L^q$ -spectrum  $\tau(\mu, q)$  of  $\mu$  can be expressed explicitly [13, 30] and has a non-differentiable point  $q_0 < 0$  [13]. In fact by using the same idea in [13], one can get the formula for  $\tau(\mu_k, q)$  and prove  $\tau(\mu_k, q) = \tau(q)$  for all  $q \in \mathbb{R}$  and  $k \in \Lambda$ . Since  $P_M(q) = (\log \rho)\tau(\mu_k, q)$  for  $q \in \mathbb{R}$ ,  $P_M(q)$  has a non-differentiable point at  $q_0$ .

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