ESTIMATES ON THE DIMENSION OF SELF-SIMILAR MEASURES WITH OVERLAPS

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Abstract. In this paper, we provide an algorithm to estimate from below the dimension of self-similar measures with overlaps. As an application, we show that for any $\beta \in (1,2)$, the dimension of the Bernoulli convolution μ_β satisfies

$$\dim(\mu_\beta) \geq 0.98040856,$$

which improves a previous uniform lower bound 0.82 obtained by Hare and Sidorov [16]. This new uniform lower bound is very close to the known numerical approximation $0.98040931953 \pm 10^{-11}$ for $\dim \mu_{\beta_3}$, where $\beta_3 \approx 1.839286755214161$ is the largest root of the polynomial $x^3 - x^2 - x - 1$. Moreover, the infimum $\inf_{\beta \in (1,2)} \dim(\mu_\beta)$ is attained at a parameter β_* in a small interval

$$(\beta_3 - 10^{-8}, \beta_3 + 10^{-8}).$$

When β is a Pisot number, we express $\dim(\mu_\beta)$ in terms of the measure-theoretic entropy of the equilibrium measure for certain matrix pressure function, and present an algorithm to estimate $\dim(\mu_\beta)$ from above as well.

1. Introduction

This paper is devoted to the dimension estimations for self-similar measures with overlaps.

Let us first introduce some notation and definitions. By an iterated function system (IFS) on $\mathbb{R}^d$ we mean a finite family $\{S_i\}_{i=1}^{\ell}$ of contracting transformations on $\mathbb{R}^d$. By Hutchinson [22], for a given IFS $\{S_i\}_{i=1}^{\ell}$ on $\mathbb{R}^d$ there is a unique nonempty compact set $K \subset \mathbb{R}^d$ such that

$$K = \bigcup_{i=1}^{\ell} S_i(K).$$

The set K is called the *attractor* of $\{S_i\}_{i=1}^{\ell}$. Moreover, K is called a *self-similar set* if all S_i are similarities, and a *self-affine set* if all S_i are affine maps.

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Let $(p_1,\ldots,p_\ell)$ be a probability vector, that is, $p_i>0$ for all i and $\sum_{i=1}^{\ell} p_i=1$. It is well-known [22] that there is a unique Borel probability measure μ on $\mathbb{R}^d$ such that

$$\mu=\sum_{i=1}^{\ell} p_i \mu\circ S_i^{-1}.$$

Moreover, μ is supported on K. We call μ the *stationary measure* associated with $\{S_i\}_{i=1}^{\ell}$ and $(p_1,\ldots,p_\ell)$. In particular, μ is said to be *self-similar* if S_i are all similarities and *self-affine* if S_i are all affine maps.

It is known that every self-similar measure μ on $\mathbb{R}^d$ is exact dimensional (see [9]), that is to say, there exists a constant C such that

$$\lim_{r\to 0}\frac{\log \mu(B_r(x))}{\log r}=C$$

for μ-a.e. $x\in\mathbb{R}^d$, where $B_r(x)$ stands for the closed ball of radius r centred at x. We write $\dim(\mu)$ for this constant and call it the *dimension* of μ.

One of the fundamental questions in fractal geometry is to determine the dimension of self-similar measures. So far this question has been well-understood when the underlying IFS satisfies the open set condition [22] or the exponential separation condition [17, 18]. However, in the general overlapping case, although there are some significant advances in recent years (see e.g. [26, 27, 29] and the survey papers [19, 28]) the question still remains wide open.

The present paper aims to provide some methods to estimate the dimension of self-similar measures from below and above. To state our result, define $\varphi:[0,\infty)\to\mathbb{R}$ by $\varphi(x)=-x\log x$. Set $\mathbb{R}_+=[0,\infty)$. For $\ell\in\mathbb{N}$, define $f_\ell:\mathbb{R}_+^\ell\to\mathbb{R}$ by

$$f_\ell(x_1,\ldots,x_\ell)=(x_1+\cdots+x_\ell)\sum_{i=1}^{\ell}\varphi\left(\frac{x_i}{x_1+\cdots+x_\ell}\right). \tag{1.1}$$

The function f_ℓ is monotone increasing on $\mathbb{R}_+^\ell$ (see Lemma 3.4).

For a Borel probability measure η on $\mathbb{R}^d$, a finite collection $\mathscr{D}$ of Borel subsets of $\mathbb{R}^d$ is said to be a *finite Borel partition of* $\mathbb{R}^d$ *with respect to* η if $\eta\left(\bigcup_{D\in\mathscr{D}} D\right)=1$ and $\eta(D\cap D')=0$ for different elements $D,D'\in\mathscr{D}$. Our first result is the following.

Theorem 1.1. *Let μ be the self-similar measure associated with an IFS $\{S_i\}_{i=1}^{\ell}$ on $\mathbb{R}^d$ and a probability vector $(p_1,\ldots,p_\ell)$. Let ρ_i denote the contraction ratio of S_i, $i=1,\ldots,\ell$. Then the following properties hold.*

(i) *For any finite Borel partition $\mathscr{D}$ of $\mathbb{R}^d$ with respect to μ,*

$$\dim(\mu)\geq\frac{\left(\sum_{i=1}^{\ell}-p_i\log p_i\right)-\sum_{D\in\mathscr{D}} f_\ell\left(p_1\mu(S_1^{-1}D),\ldots,p_\ell\mu(S_\ell^{-1}D)\right)}{\sum_{i=1}^{\ell}-p_i\log\rho_i}; \tag{1.2}$$

consequently, if $p_i\mu(S_i^{-1}D) \le y_i(D)$ *for* $1 \le i \le \ell$ *and* $D \in \mathscr{D}$, *then*

$$\dim(\mu) \ge \frac{\left(\sum_{i=1}^{\ell} -p_i \log p_i\right) - \sum_{D\in\mathscr{D}} f_\ell\left(y_1(D), \ldots, y_\ell(D)\right)}{\sum_{i=1}^{\ell} -p_i \log \rho_i}. \tag{1.3}$$

(ii) *Let* $(\mathscr{D}_n)$ *be a sequence of finite Borel partitions of* $\mathbb{R}^d$ *with respect to* μ *so that* $\max_{D\in\mathscr{D}_n} \operatorname{diam}(D) \to 0$ *as* $n \to \infty$. *Then*

$$\dim(\mu) = \frac{\left(\sum_{i=1}^{\ell} -p_i \log p_i\right) - \lim_{n\to\infty} \sum_{D\in\mathscr{D}_n} f_\ell\left(p_1\mu(S_1^{-1}D), \ldots, p_\ell\mu_\ell(S_\ell^{-1}D)\right)}{\sum_{i=1}^{\ell} -p_i \log \rho_i}.$$

The above theorem is based on a result of the first author and Hu [9] which states that the dimension of a self-similar measure can be expressed in terms of the projection entropy. The reader is referred to Section 2 for the involved notation and Theorem 2.2 for the details of this result. Theorem 1.1 then follows directly from some lower bound estimates on the projection entropy. It provides a valid way to estimate the dimension of general self-similar measures from below. Two examples (see Examples 4.1-4.2) are given to illustrate this method.

As an interesting application, we can elaborate the above method to obtain a new uniform lower bound on the dimension of Bernoulli convolutions. Recall that for each $\beta \in (1,2)$ the Bernoulli convolution with parameter β, say μ_β, is the self-similar measure associated with the IFS $\{\beta^{-1}x, \beta^{-1}x + 1 - \beta^{-1}\}$ on $\mathbb{R}$ and the probability vector $(1/2, 1/2)$. Bernoulli convolutions are one of the most studied examples of overlapping self-similar measures and they are of great interest in fractal geometry (see e.g. the survey articles [25, 28]). It is known that $\dim(\mu_\beta) = 1$ if β is a transcendental number [29] and $\dim(\mu_\beta) < 1$ if β is a Pisot number [13]. Recall that a Pisot number is an algebraic integer all of whose Galois conjugates are inside the unit disk. So far Pisot numbers in $(1,2)$ are the only known examples of parameters β with $\dim(\mu_\beta) < 1$. As for other algebraic numbers, it is known that $\dim(\mu_\beta) = 1$ if β is an algebraic number which is not a root of a polynomial of coefficients 0 and ± 1 [17], or β is an algebraic number with relatively large Mahler measure [4], or β is among some concrete examples of algebraic numbers with small degree [1, 15].

Recently, Hare and Sidorov [16] showed that $\dim(\mu_\beta) \ge 0.82$ for all $\beta \in (1,2)$. Their algorithm is based on the estimation for the maximal growth rate of overlapping times of the underlying IFS under iterations. By elaborating the algorithm in Theorem 1.1, we can provide a new uniform lower bound on the dimension of Bernoulli convolutions. To state our result, let $\beta_3 \approx 1.839286755214161$ be the tribonacci number, i.e. the largest root of the polynomial $x^3 - x^2 - x - 1$. A computable theoretical formula for $\dim \mu_{\beta_3}$ (expressed in a series) was independently obtained by the first author [5, 8] and Grabner et al. [14], with a corresponding numerical estimation

$$\dim(\mu_{\beta_3}) \approx 0.98040931953 \pm 10^{-11}. \tag{1.4}$$

Now we are ready to state our uniform lower bound.

Theorem 1.2. $\dim(\mu_\beta) \geq 0.98040856$ *for all* $\beta \in (1,2)$. *Moreover,* $\dim(\mu_\beta) > \dim(\mu_{\beta_3})$ *if*

$$\beta \in (\sqrt{2}, 2) \backslash (\beta_3 - 10^{-8},\ \beta_3 + 10^{-8}).$$

It can be proved (see Lemma 6.3) that there exists $\beta_* \in (\sqrt{2}, 2)$ such that

$$\dim(\mu_{\beta_*}) = \inf_{\beta\in(1,2)} \dim(\mu_\beta).$$

According to Theorem 1.2, $\beta_* \in (\beta_3 - 10^{-8},\ \beta_3 + 10^{-8})$. This leads to the following.

Conjecture 1.3. $\beta_* = \beta_3$. *Moreover,* $\dim(\mu_\beta) > \dim(\mu_{\beta_3})$ *if* $\beta \in (1,2)\backslash\{\beta_3\}$.

In the remaining part of this section, we turn to the question how to estimate $\dim(\mu_\beta)$ with small error when β is a Pisot number. So far, this question has only been answered in the special case when $\beta = \beta_n$, $n = 2, 3, \ldots$, where β_n is the largest root of the polynomial $x^n - x^{n-1} - x^{n-2} - \cdots - x - 1$. In such situation, there are computable theoretical formulas for $\dim(\mu_{\beta_n})$; see [2] for the case when $n = 2$ and [5, 8, 14] for the general case. This is due to the following special property of μ_{β_n}: they are (locally) self-similar measures associated with infinite IFSs with no overlaps (see e.g. [5, 8]). However, this property seems not to be generic in the Pisot cases.

In [24] Lalley showed that for each Pisot number $\beta \in (1,2)$, $\dim(\mu_\beta)$ can be expressed in terms of the top Lyapunov exponent of a sequence of random matrix products. Although Lalley provided an algorithm for the construction of these matrices, but the computation of Lyapunov exponents is a very difficult problem and Lalley only provided some numerical estimates on the dimension of the standard Bernoulli convolution (and its biased versions) associated with β_2.

Built on an early work of the first author [6] and the thermodynamic formalism for matrix products, for each Pisot number $\beta \in (1,2)$, in what follows we will express $\dim(\mu_\beta)$ in terms of the entropy of the equilibrium measure for certain matrix pressure function, and give some computable upper bounds on $\dim(\mu_\beta)$ in terms of conditional entropies.

Recall that in [6], for a given Pisot number $\beta \in (1,2)$, we can construct a finite family of $d \times d$ non-negative matrices $A_1, \ldots, A_k$, where d and k depend on β, so that

$$H := \sum_{i=1}^{k} A_i \tag{1.5}$$

is irreducible (i.e. there exists an integer p such that $\sum_{j=1}^{p} H^j$ is a strictly positive matrix), and moreover,

$$\dim(\mu_\beta) = -\frac{P'(1)}{\log\beta}, \qquad \tau_{\mu_\beta}(q) = -\frac{P(q)}{\log\beta} \quad \text{for} \quad q > 0, \tag{1.6}$$

where $P\colon (0,\infty) \to \mathbb{R}$ denotes the pressure function associated with $(A_1, \ldots, A_k)$, which is defined by

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{i_1\cdots i_n \in \{1,\ldots,k\}^n} \|A_{i_1}\cdots A_{i_n}\|^q \right), \tag{1.7}$$

here $\|\cdot\|$ is the standard matrix norm, and $\tau_{\mu_\beta}(q)$ stands for the L^q-spectrum of μ_β, which is defined by

$$\tau_{\mu_\beta}(q) = \liminf_{r\to 0} \frac{\log \sum_{Q\in\mathcal{Q}_r} \mu_\beta(Q)^q}{\log r}, \qquad q > 0,$$

where $\mathcal{Q}_r := \{[jr, (j+1)r) : \ j \in \mathbb{Z}\}$. See [6, Theorems 1.3 and 5.2]. By definition $\tau_{\mu_\beta}(1) = 0$, it follows from (1.6) that $P(1) = 0$ which implies that 1 is the largest eigenvalue of H. Since H is an irreducible non-negative matrix, it has left and right positive eigenvectors associated to the eigenvalue 1 (see e.g. [21, Theorem 8.4.4]). Let $\mathbf{v}_L$, $\mathbf{v}_R$ be the left and right positive eigenvectors of H such that $\mathbf{v}_L \cdot \mathbf{v}_R = 1$. Define a Borel probability measure η on $\Sigma = \{1, \ldots, k\}^{\mathbb{N}}$ by

$$\eta\left([i_1\cdots i_n]\right) = \mathbf{v}_L A_{i_1}\cdots A_{i_n}\mathbf{v}_R \quad \text{for} \ \ n\in\mathbb{N} \ \ \text{and} \ \ i_1\cdots i_n \in \{1,\ldots,k\}^n, \tag{1.8}$$

where $[i_1\cdots i_n] := \left\{x = (x_j)_{j=1}^\infty \in \Sigma : \ x_j = i_j \text{ for } 1\le j\le n\right\}$.

It is readily checked that η is indeed a probability measure and moreover, η is σ-invariant, where $\sigma\colon \Sigma\to\Sigma$ is the left shift map defined by $(x_j)_{j=1}^\infty \mapsto (x_{j+1})_{j=1}^\infty$. Let $h_\eta(\sigma)$ denote the measure-theoretic entropy of η with respect to σ (see [30] for a definition). Now we can state our last result.

Theorem 1.4. *Let $\beta\in(1,2)$ be a Pisot number, and let η be constructed as above. Then $\dim(\mu_\beta) = \dfrac{h_\eta(\sigma)}{\log\beta}$. Consequently, for each $n\in\mathbb{N}$,*

$$\dim(\mu_\beta) \le \frac{\sum_{I\in\{1,\ldots,k\}^{n+1}} \varphi\left(\eta([I])\right) - \sum_{J\in\{1,\ldots,k\}^{n}} \varphi\left(\eta([J])\right)}{\log\beta}, \tag{1.9}$$

where $\varphi(x) = -x\log x$.

The inequality (1.9) provides a sequence of upper bounds on $\dim(\mu_\beta)$ for a general Pisot number β. When β is of small degree, the dimension and the number of the constructed matrices A_i are not very large, so one can use (1.9) to obtain reasonable upper bounds on $\dim(\mu_\beta)$. In the meantime one can use the algorithm in Theorem 1.1 to estimate $\dim(\mu_\beta)$ from below. Hence in this situation one can provide the estimates on $\dim(\mu_\beta)$ with small error. In Section 7, we list our computational results on $\dim(\mu_\beta)$ for some Pisot numbers of degree 3 or 4. For instance, let $\beta \approx 1.465571231876768$ be the largest root of the polynomial $x^3 - x^2 - 1$, our computation shows that $\dim(\mu_\beta) \approx 0.99954470$ with an error $\leq 10^{-8}$.

It is worth pointing out that the method of estimating projection entropies can also be used to find lower bounds on the dimension of self-affine measures generated by diagonal affine iterated function systems. In Section 8, we will provide a more detailed justification about this fact and give an example.

The main methods of this paper (together with a previously obtained uniform lower bound 0.980368 on $\dim \mu_\beta$) were announced by the first author in the conference "Number theory and dynamics" at Cambridge in March 2019. Before completing the writing of this paper, we were aware of a very recent work [23] by Kleptsyn, Pollicott and Vytnova, who obtained a uniform lower bound 0.96399 on the Hausdorff dimension of Bernoulli convolutions through a different approach by estimating the L^2-dimension of Bernoulli convolutions from below.

The paper is organised as follows: In Section 2 we give the definition of projection entropy and present a result in [9] on the dimension of self-similar measures. In Section 3, we give some upper bound estimates on conditional entropies and prove Theorem 1.1. In Section 4, we give an example to illustrate the application of Theorem 1.1. In Section 5, we provide an algorithm to produce uniform lower bounds on $\dim(\mu_\beta)$ over small intervals of β, and give a computer-assisted proof of Theorem 1.2. In Section 6, we investigate the asymptotic properties of $\dim(\mu_\beta)$ when β is close to 2. In Section 7, we prove Theorem 1.4 and give computational results on $\dim(\mu_\beta)$ for some Pisot numbers of degree 3 or 4. In Section 8, we give some final remarks.

2. Preliminary

In this section, we introduce the concept of projection entropy for a Bernoulli product measure associated with an IFS and present a result in [9] on the dimension of self-similar measures.

Let $\{S_i\}_{i=1}^{\ell}$ be an IFS on $\mathbb{R}^d$ with attractor K, and let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$. That is, $\Sigma = \{1, \ldots, \ell\}^{\mathbb{N}}$ and σ is the left shift on Σ defined by

$$\sigma((x_n)_{n=1}^{\infty}) = (x_{n+1})_{n=1}^{\infty}.$$

Let $\pi : \Sigma \to \mathbb{R}^d$ be the canonical coding map associated with the IFS $\{S_i\}_{i=1}^{\ell}$. That is,

$$\pi(x) = \lim_{n\to\infty} S_{x_1} \circ \cdots \circ S_{x_n}(0), \qquad x = (x_n)_{n=1}^{\infty}.$$

Let $m = \prod_{n=1}^{\infty}\{p_1, \dots, p_\ell\}$ be the Bernoulli product measure on Σ and $\mu = m \circ \pi^{-1}$, that is,

$$\mu(A) = m(\pi^{-1}(A))$$

for any Borel subset A of $\mathbb{R}^d$. Clearly, μ is supported on K. It is well-known ([22]) that μ is the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\mu = \sum_{i=1}^{\ell} p_i \mu \circ S_i^{-1}. \tag{2.1}$$

Let $\mathcal{P} = \{[i] \colon i = 1, \dots, \ell\}$ be the natural partition of Σ, where

$$[i] := \{x = (x_n)_{n=1}^{\infty} \in \Sigma : \; x_1 = i\}.$$

The following definition was introduced in [9].

Definition 2.1. The projection entropy of m under π is

$$h_\pi(\sigma, m) := H_m(\mathcal{P}) - H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)),$$

where $H_m(\mathcal{P}) = \sum_{i=1}^{\ell} -p_i \log p_i$ is the entropy of the partition $\mathcal{P}$, and $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$ and $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$ is the conditional entropy of $\mathcal{P}$ given $\pi^{-1}\mathcal{B}(\mathbb{R}^d)$.

The reader is referred to [30] for the definition of conditional entropy. The concept of projection entropy plays a crucial role in the dimension theory of IFS [9]. In particular, it can be used to characterize the dimension of self-similar measures.

Theorem 2.2 ([9, Theorem 2.8])**.** *Let $\{S_i\}_{i=1}^{\ell}$ be an IFS consisting of similarities. Let ρ_i be the contraction ratio of S_i, $i = 1, \dots, \ell$. Then μ is exact dimensional and*

$$\dim(\mu) = \frac{h_\pi(m)}{\lambda} = \frac{H_m(\mathcal{P}) - H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))}{\lambda}$$

where $\lambda := -\sum_{i=1}^{\ell} p_i \log \rho_i$.

Since $H_m(\mathcal{P})$ and λ are easy to compute, in order to estimate $\dim_H(\mu)$, it is sufficient to estimate $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$.

In the next section, we provide an algorithm to estimate $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$ from above for any given IFS on $\mathbb{R}^d$.

3. Upper bounds on $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$ and the proof of Theorem 1.1

Let $\{S_i\}_{i=1}^{\ell}$ be an IFS on $\mathbb{R}^d$ with attractor K. Here we only assume that S_i are contracting. Let π, m, μ and $\mathcal{P}$ be defined as in Section 2. Recall that for a finite Borel partition $\mathcal{A}$ of Σ,

$$H_m(\mathcal{P}|\mathcal{A}) := H_m(\mathcal{P} \vee \mathcal{A}) - H_m(\mathcal{A}), \tag{3.1}$$

where $\mathcal{P} \vee \mathcal{A} := \{P \cap A \colon P \in \mathcal{P},\ A \in \mathcal{A}\}$; see e.g. [30, §4.3].

In this section we will provide upper bounds on the condition entropy $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$, and prove Theorem 1.1. In the end of this section, we also give an iterative algorithm to estimate $\mu(A)$ from above with small error for a given bound Borel set $A \subset \mathbb{R}^d$.

The following result is our starting point.

Lemma 3.1. (i) *Let $\mathscr{D}$ be a finite Borel partition of $\mathbb{R}^d$. Then*

$$H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) \le H_m(\mathcal{P}|\pi^{-1}\mathscr{D}).$$

(ii) *Let $\{\mathscr{D}_n\}$ be a sequence of finite Borel partitions of K with*

$$\operatorname{diam}(\mathscr{D}_n) := \sup_{D\in\mathscr{D}_n} \operatorname{diam}(D) \to 0.$$

Then $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) = \lim_{n\to\infty} H_m(\mathcal{P}|\pi^{-1}\mathscr{D}_n)$.

Proof. Let $\mathscr{D}$ be a finite Borel partition of $\mathbb{R}^d$. Then the σ-algebra generated by $\pi^{-1}\mathscr{D}$ is a sub-σ-algebra of $\pi^{-1}\mathcal{B}(\mathbb{R}^d)$. By [30, Theorem 4.3(v)],

$$H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) \le H_m(\mathcal{P}|\pi^{-1}\mathscr{D}),$$

This proves (i).

To prove (ii), let $\{\mathscr{D}_n\}$ be a sequence of finite Borel partitions of K with $\operatorname{diam}(\mathscr{D}_n) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. In view of (i), it suffices to show that

$$\limsup_{n\to\infty} H_m(\mathcal{P}|\pi^{-1}\mathscr{D}_n) \le H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) + \varepsilon. \tag{3.2}$$

To this end, take a sequence $\{\mathscr{A}_n\}$ of finite Borel partitions of K such that

$$\mathscr{A}_{n+1} \ge \mathscr{A}_n, \quad \operatorname{diam}(\mathscr{A}_n) \to 0, \quad \text{and} \quad \mathscr{A}_n \uparrow \mathcal{B}(K),$$

where $\mathscr{A} \ge \mathscr{C}$ means that any element in $\mathscr{C}$ is the union of elements in $\mathscr{A}$, and $\mathcal{B}(K)$ stands for the σ-algebra of Borel subsets of K. Since the range of π is K, it follows that

$$\sigma(\pi^{-1}\mathscr{A}_n) \uparrow \pi^{-1}\mathcal{B}(K) = \pi^{-1}\mathcal{B}(\mathbb{R}^d).$$

Hence by [30, Theorem 4.7], $H_m(\mathcal{P}|\pi^{-1}\mathscr{A}_n) \downarrow H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$. Take a large positive integer l such that

$$H_m(\mathcal{P}|\pi^{-1}\mathscr{A}_l) \le H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) + \varepsilon.$$

Write $\mathscr{A}_l =: \mathscr{A} = \{A_1, \ldots, A_M\}$.

Since $\operatorname{diam}(\mathscr{D}_n) \to 0$ as $n \to \infty$, by [3, Lemma 1.23] there are partitions $\mathscr{E}_n = \{E_1^n, \ldots, E_M^n\}$ of K such that

(a) Each E_i^n is a union of members of $\mathscr{D}_n$;
(b) $\lim_{n\to\infty} \mu(E_i^n \triangle A_i) = 0$ for each $1 \le i \le M$, where $A \triangle B := (A\backslash B) \cup (B\backslash A)$.

Notice that for any $1 \leq j \leq \ell$ and $1 \leq i \leq M$,

$$\begin{aligned} m(([j] \cap \pi^{-1}(E_i^n)) \triangle ([j] \cap \pi^{-1}(A_i))) &\leq m(\pi^{-1}(E_i^n) \triangle \pi^{-1}(A_i)) \\ &= \mu(E_i^n \triangle A_i) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} H_m(\mathcal{P}|\pi^{-1}\mathscr{E}_n) = H_m(\mathcal{P}|\pi^{-1}\mathscr{A}) \leq H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) + \varepsilon.$$

Since $\mathscr{D}_n \geq \mathscr{E}_n$, by [30, Theorem 4.3(v)] we have $H_m(\mathcal{P}|\pi^{-1}\mathscr{D}_n) \leq H_m(\mathcal{P}|\pi^{-1}\mathscr{E}_n)$, therefore

$$\limsup_{n\to\infty} H_m(\mathcal{P}|\pi^{-1}\mathscr{D}_n) \leq \lim_{n\to\infty} H_m(\mathcal{P}|\pi^{-1}\mathscr{E}_n) \leq H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) + \varepsilon.$$

This proves (3.2) and we are done. $\square$

Remark 3.2. Let $\mathscr{D}$ be a finite Borel partition of $\mathbb{R}^d$ with respect to μ (cf. Section 1). Since μ is supported on K, there exists a finite Borel partition $\mathscr{D}'$ of K such that for each $D \in \mathscr{D}$ there exists $D' \in \mathscr{D}'$ so that $\mu(D \triangle D') = 0$. This implies that $H_m(\mathcal{P}|\pi^{-1}\mathscr{D}) = H_m(\mathcal{P}|\pi^{-1}\mathscr{D}')$. It follows from Lemma 3.1(i) that $H_m(\mathcal{P}|\pi^{-1}\mathscr{D})$ is an upper bound of $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$.

Below we discuss how to estimate $H_m(\mathcal{P}|\pi^{-1}\mathscr{D})$ for a given finite Borel partition $\mathscr{D}$ of $\mathbb{R}^d$ with respect to μ.

Lemma 3.3. *For $D \in \mathcal{B}(\mathbb{R}^d)$ and $i \in \{1, \dots, \ell\}$,*

$$m([i] \cap \pi^{-1}(D)) = p_i \mu(S_i^{-1}(D)).$$

Proof. Observe that for $x = (x_n)_{n=1}^\infty \in \Sigma$,

$$\begin{aligned} x \in [i] \cap \pi^{-1}(D) &\iff x_1 = i,\ \pi x \in D \\ &\iff x_1 = i,\ S_i(\pi\sigma x) \in D \\ &\iff x_1 = i,\ x \in \sigma^{-1}\pi^{-1}(S_i^{-1}D) \\ &\iff x \in [i] \cap \sigma^{-1}\pi^{-1}(S_i^{-1}D). \end{aligned}$$

Hence $[i] \cap \pi^{-1}(D) = [i] \cap \sigma^{-1}\pi^{-1}(S_i^{-1}D)$. It follows that

$$\begin{aligned} m([i] \cap \pi^{-1}(D)) &= m([i] \cap \sigma^{-1}\pi^{-1}(S_i^{-1}D)) \\ &= p_i m(\pi^{-1}(S_i^{-1}D)) \\ &= p_i \mu(S_i^{-1}D), \end{aligned}$$

where in the second equality we used the property that $m([i] \cap \sigma^{-1}A) = p_i m(A)$ for any Borel subset A of Σ. $\square$

Let $f_\ell : [0, \infty)^\ell \to \mathbb{R}$ be defined as in (1.1).

Lemma 3.4. *For every $1 \le i \le \ell$, $\frac{\partial f_\ell}{\partial x_i} \ge 0$. Consequently, f_ℓ is monotone increasing over $\mathbb{R}_+^\ell$ in the sense that*

$$f_\ell(x_1+\varepsilon_1, \dots, x_\ell+\varepsilon_\ell) \ge f_\ell(x_1, \dots, x_\ell)$$

for any $x_1, \dots, x_\ell, \varepsilon_1, \dots, \varepsilon_\ell \ge 0$.

Proof. A direct computation shows that for each i,

$$\frac{\partial f_\ell}{\partial x_i}(x_1, \dots, x_\ell) = \log(x_1 + \cdots + x_\ell) - \log x_i \ge 0,$$

from which we obtain the desired inequality for f_ℓ. □

Lemma 3.5. *Let $\mathscr{D}$ be a finite partition of $\mathbb{R}^d$ with respect to μ. Then*

$$H_m(\mathcal{P}|\pi^{-1}\mathscr{D}) = \sum_{D \in \mathscr{D}} f_\ell(\mu_1(D), \dots, \mu_\ell(D)),$$

where $\mu_i(D) := p_i \mu(S_i^{-1}D)$.

Proof. By (3.1),

$$\begin{aligned} H_m(\mathcal{P}|\pi^{-1}\mathscr{D}) &= H_m(\mathcal{P} \vee \pi^{-1}\mathscr{D}) - H_m(\pi^{-1}\mathscr{D}) \\ &= \sum_{D \in \mathscr{D}} \left(\left(\sum_{i=1}^{\ell} \varphi(m([i] \cap \pi^{-1}D)) \right) - \varphi(m(\pi^{-1}D)) \right) \\ &= \sum_{D \in \mathscr{D}} \left(\left(\sum_{i=1}^{\ell} \varphi(p_i \mu(S_i^{-1}D)) \right) - \varphi(\mu(D)) \right) \\ &= \sum_{D \in \mathscr{D}} f_\ell(\mu_1(D), \dots, \mu_\ell(D)), \end{aligned}$$

where we used Lemma 3.3 in the third equality, and the fact that $\mu(D) = \sum_{i=1}^{\ell} \mu_i(D)$ (which follows from (2.1)) in the fourth equality. □

Corollary 3.6. *Let $\mathscr{D}$ be a finite partition of $\mathbb{R}^d$ with respect to μ. Suppose that*

$$p_i \mu(S_i^{-1}D) \le y_i(D) \quad \text{for every } i \in \{1, \dots, \ell\} \text{ and } D \in \mathscr{D}.$$

Then $H_m(\mathcal{P}|\pi^{-1}\mathscr{D}) \le \sum_{D \in \mathscr{D}} f_\ell(y_1(D), \dots, y_\ell(D))$.

Proof. It follows immediately from Remark 3.2, Lemmas 3.4 and 3.5. □

Now we are ready to prove Theorem 1.1.

Theorem 1.1. Part (i) follows from Theorem 2.2, Remark 3.2 and Lemma 3.5. Part (ii) follows from Theorem 2.2, Lemma 3.1, Remark 3.2 and Lemma 3.5. □

Lemma 3.1 (also Remark 3.2) and Corollary 3.6 provide us the following theoretical way to estimate $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$ from above: first choose a finite Borel partition $\mathscr{D}$ of $\mathbb{R}^d$ with respect to μ, and find $y_i(D) \geq p_i\mu(S_i^{-1}(D))$ for each $i \in \{1, \ldots, \ell\}$ and $D \in \mathscr{D}$. Then

$$H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) \leq \sum_{D\in\mathscr{D}} f_\ell(y_1(D), \ldots, y_\ell(D)). \tag{3.3}$$

In the remaining part of this section, we discuss how to find $y_i(D) \geq p_i\mu(S_i^{-1}(D))$ with $y_i(D) - p_i\mu(S_i^{-1}(D))$ being small. This reduces to the following.

Problem 3.7. *How to estimate $\mu(A)$ from above with small error for a given bounded Borel set $A \subset \mathbb{R}^d$?*

In what follows, we give an answer to the above problem by providing a simple iterative algorithm.

Algorithm 3.8. Let $A \subset \mathbb{R}^d$ be a given bounded Borel set. Take $\gamma > 0$ and a positive integer L. Choose a closed ball $B \subset \mathbb{R}^d$ such that $S_i(B) \subset B$, $i = 1, \ldots, \ell$. Then $K \subset B$. Below we inductively construct a finite sequence $(u_n)_{n=0}^L$ of non-negative numbers and a finite sequence $(\Lambda_n)_{n=0}^L$ of finite "subsets" of $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^d)$. Here for convenience, we allow that the elements in Λ_n may be the same, but they are counted as different elements.

(1) Set $u_0 = 0$ and $\Lambda_0 = \{(1, A)\}$.

(2) Define
$$\Lambda_1^* = \bigcup_{(t,E)\in\Lambda_0} \left\{\left(tp_i + \gamma, V_\gamma(S_i^{-1}E) \cap B\right) :\ i = 1, \ldots, \ell\right\}.$$
where $V_\gamma(E) := \{x\colon d(x, E) < \gamma\}$ for $E \neq \emptyset$, and $V_\gamma(\emptyset) = \emptyset$. Write $\Lambda_1^* = \{(t_j, E_j) :\ j = 1, \ldots, k_1\}$. Keep in mind that we don't require that (t_j, E_j) are distinct for different j. For $j = 1, \ldots, k_1$, we apply the following operations consecutively:
  (a) If $E_j = \emptyset$, then remove the element (t_j, E_j) from Λ_1^*.
  (b) If $E_j \supset B$, then remove the element (t_j, E_j) from Λ_1^*, and add t_j to u_0.

Finally set $u_1 = u_0$ and $\Lambda_1 = \Lambda_1^*$.

(3) Suppose we have obtained u_n and Λ_n for some $n < N$. Similar to Step (2), we set
$$\Lambda_{n+1}^* = \bigcup_{(t,E)\in\Lambda_n} \left\{\left(tp_i + \gamma, V_\gamma(S_i^{-1}E) \cap B\right) :\ i = 1, \ldots, \ell\right\},$$
and write it as $\Lambda_{n+1}^* = \{(t_j, E_j) :\ j = 1, \ldots, k_{n+1}\}$. For $j = 1, \ldots, k_{n+1}$, we apply the following operations consecutively:
  (a) If $E_j = \emptyset$, then remove the element (t_j, E_j) from Λ_{n+1}^*.
  (b) If $E_j \supset B$, then remove the element (t_j, E_j) from Λ_{n+1}^*, and add t_j to u_n.

Finally set $u_{n+1} = u_n$ and $\Lambda_{n+1} = \Lambda^*_{n+1}$.

(4) Repeating the above procedures until we obtain u_L and Λ_L. Finally let

$$u^* = u_L + \sum_{(t,E)\in\Lambda_L} t,$$

Then u^* is an upper bound for $\mu(A)$.

Remark 3.9. The introduction of γ in the above algorithm is used to compensate the computation error, and it should be selected according to the computation precision. The number L in the algorithm is called the iteration time.

4. TWO EXAMPLES

In this section, we use two examples to illustrate how to apply Theorem 1.1 to get lower bound estimates on the dimension of self-similar measures.

Example 4.1. Let μ be the self-similar measure associated with an IFS $\{S_i\}_{i=1}^3$ on $\mathbb{R}$ and the probability vector $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, where

$$S_1(x) = \frac{1}{2}x,\ S_2(x) = \frac{x+1}{3},\ S_3(x) = \frac{x+3}{4}.$$

Let K be the attractor of $\{S_i\}_{i=1}^3$. It is easily checked that the convex hull of K is the interval $[0,1]$. Let μ be the self-similar measure associated with $\{S_i\}_{i=1}^3$ and the probability vector $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$. Let $m = \prod_{n=1}^\infty \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ be the Bernoulli product measure on $\Sigma = \{1,2,3\}^{\mathbb{N}}$. Let

$$\mathscr{D}_1 = \left\{ \left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right] \right\},$$

which is the partition of $[0,1]$ (with respect to μ) generated by the endpoints of the intervals $S_i[0,1], i = 1,2,3$. This IFS is somehow special, in which the contraction ratio of each map is the reciprocal of an integer, and the translation part is a rational number. Due to this property, we can use the self-similarity of μ to compute the precise value of $\mu(S_i^{-1}D)$ for every i and $D \in \mathscr{D}_1$. To see this, applying the self-similarity relation $\mu = \sum_{i=1}^3 3^{-1}\mu \circ S_i^{-1}$ to the interval $[0,1/3]$, we get

$$\begin{aligned}\mu([0,1/3]) &= 3^{-1}\mu([0,2/3]) + 3^{-1}\mu([-1,0]) + 3^{-1}\mu([-3,-5/3])\\ &= 3^{-1}\mu([0,2/3]),\end{aligned}$$

where in the second equality we used the facts that μ is supported on $[0,1]$ and has no atoms. Similarly, we have

$$\begin{aligned}\mu([0,2/3]) &= 3^{-1}\mu([0,4/3]) + 3^{-1}\mu([-1,1]) + 3^{-1}\mu([-3,-1/3])\\ &= 2/3.\end{aligned}$$

$D \in \mathscr{D}_1$	$\mu(S_1^{-1}D)$	$\mu(S_2^{-1}D)$	$\mu(S_3^{-1}D)$
$[0, 1/3]$	$2/3$	0	0
$[1/3, 1/2]$	$1/3$	$1/2$	0
$[1/2, 2/3]$	0	$1/2$	0
$[2/3, 3/4]$	0	0	0
$[3/4, 1]$	0	0	1

TABLE 1. Precise values of $\mu(S_i^{-1}D)$ for $i = 1, 2, 3$ and $D \in \mathscr{D}_1$.

From the above equalities, we see that $\mu(S_1^{-1}[0, 1/3]) = 2/3$ and $\mu(S_2^{-1}[0, 1/3]) = \mu(S_3^{-1}[0, 1/3]) = 0$. Similarly by direct computations with a bare hand we can obtain the precise value of $\mu(S_i^{-1}D)$ for each $i \in \{1, 2, 3\}$ and $D \in \mathscr{D}_1$; see Table 1.

It follows that

$$\sum_{D\in\mathscr{D}_1} f_3\left(3^{-1}\mu(S_1^{-1}D), 3^{-1}\mu(S_2^{-1}D), 3^{-1}\mu(S_3^{-1}D)\right) = \frac{5}{18}\left(\varphi(\frac{2}{5}) + \varphi(\frac{3}{5})\right). \tag{4.1}$$

By Theorem 1.1(i),

$$\begin{aligned}\dim(\mu) &\geq \frac{\log 3 - \frac{5}{18}\left(\varphi(\frac{2}{5}) + \varphi(\frac{3}{5})\right)}{\frac{1}{3}\log 2 + \frac{1}{3}\log 3 + \frac{1}{3}\log 4} \\ &\approx 0.86058762883316.\end{aligned}$$

Alternatively, instead of computing the precise values of $\mu(S_i^{-1}D)$ we may use Algorithm 3.8 to estimate $\mu(S_i^{-1}D)$ from above, then use (1.3) in Theorem 1.1(i) to get a lower bound of $\dim(\mu)$.

In the above computation, if we replace $\mathscr{D}_1$ by $\mathscr{D}_n$ ($n \geq 2$), where $\mathscr{D}_n$ is the partition of $[0, 1]$ generated by the endpoints of the intervals $S_{i_1\ldots i_n}([0, 1])$, $i_1 \ldots i_n \in \{1, 2, 3\}^n$, then we can obtain larger lower bounds on $\dim(\mu)$. In practice, for $n = 2, 3, 4, 5, 6$, we manage to run a program to compute the precise value of $\mu(S_i^{-1}D)$ for each $1 \leq i \leq 3$ and $D \in \mathscr{D}_n$, and use (1.2) to get the corresponding lower bounds on $\dim(\mu)$. Whilst for $7 \leq n \leq 14$, we use Algorithm 3.8 (in which we take $L = 40$) to estimate $\mu(S_i^{-1}D)$ from above for each i and $D \in \mathscr{D}_n$, and use (1.3) to get the corresponding lower bounds. In Table 2 we list our computational results.

Example 4.2. Let $\mu = \mu_{\beta_3}$ be the Bernoulli convolution with parameter β_3, where $\beta_3 \approx 1.83928675521416$ is the tribonacci number. For $n \geq 1$, let $\mathscr{D}_n$ be the partition of $[0, 1]$ generated by the endpoints of the intervals $S_{i_1\ldots i_n}([0, 1])$, $i_1 \ldots i_n \in \{1, 2\}^n$, where $S_1(x) = x/\beta_3$, $S_2(x) = x/\beta_3 + 1 - 1/\beta_3$. Similar to Example 4.1, we can use Theorem 1.1 (in which we take $\mathscr{D} = \mathscr{D}_n$) to obtain the lower bounds on $\dim \mu_{\beta_3}$ by either computing the precise values of $\mu(S_i^{-1}D)$, or by estimating $\mu(S_i^{-1}D)$ from

Partition level n	Lower bound on $\dim(\mu)$
1	0.86058762883316
2	0.873884695870383
3	0.887965887736415
4	0.901645728024083
5	0.909541991955753
6	0.915681937458243
7	0.920399986771506
8	0.924018201523078
9	0.926957262754457
10	0.929374519513162
11	0.931389937165221
12	0.933105444767198
13	0.934566254269004
14	0.935825938794224

TABLE 2. Lower bounds on the dimension of μ in Example 4.1

Partition level n	Lower bound (I)	Lower bound (II)
1	0.974971672609929	0.974971672566547
2	0.974971672609929	0.974971672568187
3	0.974971672609929	0.974971672570255
4	0.979950375568122	0.979950375495215
5	0.979950375568122	0.979950375438316
6	0.979950375568122	0.979950375332795
7	0.980368793386354	0.980368792874582
8	0.980368793386354	0.980368792346810
9	0.980368793386354	0.980368791273832
10	0.980405622363758	0.980405618127927
11	0.980405622363758	0.980405614052234
12	0.980405622363758	0.980405606355333
13	0.980408973316171	0.980408941535664
14	0.980408973316170	0.980408909920482

TABLE 3. Lower bounds on the dimension of $\mu = \mu_{\beta_3}$ in Example 4.2, using two different methods for the evaluation of $\mu(S_i^{-1}D)$.

above via Algorithm 3.8 (in which we take $L = 40$). In Table 3, we list our computational results, where the values in the second column are obtained by using the first approach, and that in the third column are obtained by using the second approach.

5. A uniform lower bound on the dimension of Bernoulli convolutions

This section is concerned with a computer-assisted proof of Theorem 1.2.

For $\beta > 1$, let μ_β be the Bernoulli convolution associated with β. This is, μ_β is the self-similar measure associated with the IFS

$$\left\{S_{1,\beta}(x) = \beta^{-1}x,\ S_{2,\beta}(x) = \beta^{-1}x + 1 - \beta^{-1}\right\}$$

and the probability vector $(\frac{1}{2}, \frac{1}{2})$.

Let us begin with an elementary result.

Lemma 5.1. (i) *For each $\beta > 1$ and $k \in \mathbb{N}$, we have $\dim(\mu_\beta) \geq \dim_H(\mu_{\beta^k})$. Consequently if $\beta^k \geq 2$, then*

$$\dim(\mu_\beta) \geq \frac{\log 2}{k \log \beta}.$$

(ii) *For $\beta \in [\sqrt{2},\ 1.424041]$,*

$$\dim(\mu_\beta) \geq 0.98041 > \dim(\mu_{\beta_3}).$$

Proof. Part(i) was proved in [16, Proposition 2.1] for algebraic parameter values β. The extension to the general parameters is similar in spirit. For completeness, we include a proof. Let ν_β denote the probability distribution of the random series

$$\sum_{n=0}^{\infty} \epsilon_n \beta^{-n}, \tag{5.1}$$

where (ϵ_n) is a sequence of independent and identically distributed random variables, taking the values 0 and 1 with equal probability. It is easy to check that $\mu_\beta(\cdot) = \nu_\beta\left(\frac{\beta}{\beta-1}\cdot\right)$, so $\dim(\mu_\beta) = \dim(\nu_\beta)$. Meanwhile, we note that for $k \in \mathbb{N}$,

$$\nu_\beta = \nu_{\beta^k} * \eta$$

for some probability measure η. To see this decomposition, consider the series (5.1) and separate the terms divisible by k from the rest. Hence by [11, Lemma 2.2], we have

$$\dim(\nu_\beta) = \dim(\nu_{\beta^k} * \eta) \geq \dim(\nu_{\beta^k}).$$

So $\dim(\mu_\beta) \geq \dim(\mu_{\beta^k})$. Whenever $\beta^k \geq 2$, the IFS $\{S_{1,\beta^k}, S_{2,\beta^k}\}$ satisfies the open set condition, it follows that $\dim(\mu_{\beta^k}) = \log 2/(k \log \beta)$. This proves (i).

To see (ii), let $\beta \in [\sqrt{2},\ 1.424041]$. Then $\beta^2 \geq 2$, so by (i),

$$\dim(\mu_\beta) \geq \frac{\log 2}{2 \log \beta} \geq \frac{\log 2}{2 \log 1.424041} \approx 0.980410065731842 > \dim(\mu_{\beta_3}),$$

where in the last inequality we used (1.4). This completes the proof. ☐

Next we present our main method for producing a uniform lower bound on $\dim(\mu_\beta)$ when β runs over $(\sqrt{2}, 2)$. For given $\beta > 1$ and $N \in \mathbb{N}$, let $\mathscr{D}_{N,\beta}$ be the partition of $[0,1]$ generated by the points in the following set

$$\bigcup_{I\in\{1,2\}^N} \{S_{I,\beta}(0),\ S_{I,\beta}(1)\}, \tag{5.2}$$

where $S_{I,\beta} := S_{i_1,\beta} \circ \cdots \circ S_{i_N,\beta}$ for $I = i_1 \cdots i_N$. Let $f_2 : \mathbb{R}_+^2 \to \mathbb{R}$ be defined as in (1.1) (in which we take $\ell = 2$).

Proposition 5.2. *Let $\beta \in [\sqrt{2}, 2)$ and $\delta > 0$. Set*

$$\epsilon := \epsilon(\beta, \delta) = \begin{cases} \frac{\delta}{\beta}(1 + \frac{3}{\beta^4}) & \text{if } \beta < 1.5, \\ \frac{\delta}{\beta}(1 + \frac{2}{\beta^3}) & \text{if } \beta \geq 1.5. \end{cases} \tag{5.3}$$

For $N \in \mathbb{N}$, set

$$t(\beta, \delta, N) = \sum_{[a,b]\in\mathscr{D}_{N,\beta}} f_2\left(\frac{1}{2}\mu_\beta(S_{1,\beta}^{-1}[a-\epsilon, b+\epsilon]),\ \frac{1}{2}\mu_\beta(S_{2,\beta}^{-1}[a-\epsilon, b+\epsilon])\right).$$

Then for any $\beta' \in [\beta, \beta + \delta]$ with $\beta' \leq 2$,

$$\dim(\mu_{\beta'}) \geq \frac{(\log 2) - t(\beta, \delta, N)}{\log(\beta + \delta)}. \tag{5.4}$$

To prove the above proposition, we need several lemmas.

Lemma 5.3. *Let $\mathbf{i} = (i_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}$. Define $g_{\mathbf{i}} :\ (0,1) \to \mathbb{R}$ by*

$$g_{\mathbf{i}}(x) := (1-x)\sum_{n=1}^\infty i_n x^{n-1}, \qquad x \in (0,1). \tag{5.5}$$

Then for any positive integer $k \geq 2$ and $x \in \left[1 - \frac{1}{k}, 1 - \frac{1}{k+1}\right)$,

$$|g_{\mathbf{i}}'(x)| \leq kx^{k-1}.$$

Proof. Let $k \geq 2$ and $x \in \left[1 - \frac{1}{k}, 1 - \frac{1}{k+1}\right)$. Then

$$g_{\mathbf{i}}'(x) = -i_1 + \sum_{n=2}^\infty i_n((n-1)x^{n-2} - nx^{n-1}) = I_1 + I_2,$$

where

$$I_1 := -i_1 + \sum_{n=2}^k i_n((n-1)x^{n-2} - nx^{n-1}),$$

$$I_2 := \sum_{n=k+1}^\infty i_n((n-1)x^{n-2} - nx^{n-1}).$$

Clearly, $I_1 \leq 0$ and $I_2 > 0$. Moreover,

$$\begin{aligned} -I_1 &= i_1 + \sum_{n=2}^{k} i_n(nx^{n-1} - (n-1)x^{n-2}) \\ &\leq 1 + \sum_{n=2}^{k}(nx^{n-1} - (n-1)x^{n-2}) = kx^{k-1}, \end{aligned}$$

and

$$I_2 \leq \sum_{n=k+1}^{\infty} ((n-1)x^{n-2} - nx^{n-1}) = kx^{k-1}.$$

Hence $|g_{\mathbf{i}}(x)| = |I_1 + I_2| \leq \max\{-I_1, I_2\} \leq kx^{k-1}$. □

For $\beta > 1$, let $\pi_\beta : \{1,2\}^{\mathbb{N}} \to \mathbb{R}$ be the coding map associated with the IFS $\{S_{1,\beta},\ S_{2,\beta}\}$. A direct calculation yields that

$$\pi_\beta(x) = (1 - \beta^{-1}) \sum_{n=1}^{\infty}(x_n - 1)\beta^{-(n-1)}, \qquad x = (x_n)_{n=1}^{\infty}. \tag{5.6}$$

Lemma 5.4. *Let $\sqrt{2} \leq \beta < \beta' \leq 2$. Then*

(i) *For any $u \in \{1,2\}^{\mathbb{N}}$,*

$$|\pi_\beta(u) - \pi_{\beta'}(u)| \leq \begin{cases} 2\beta^{-3}(\beta' - \beta) & \text{if } \beta > 1.5, \\ 3\beta^{-4}(\beta' - \beta) & \text{if } \beta \leq 1.5. \end{cases}$$

(ii) *For $c, d \in \mathbb{R}$ with $c < d$,*

$$\mu_{\beta'}([c,\ d]) \leq \begin{cases} \mu_\beta([c - 2\beta^{-3}(\beta' - \beta),\ d + 2\beta^{-3}(\beta' - \beta)]) & \text{if } \beta > 1.5, \\ \mu_\beta([c - 3\beta^{-4}(\beta' - \beta),\ d + 3\beta^{-4}(\beta' - \beta)]) & \text{if } \beta \leq 1.5. \end{cases}$$

Proof. We first prove (i). Let $u = (u_n)_{n=1}^{\infty} \in \{1,2\}^{\mathbb{N}}$. Define $\mathbf{i} = (i_n)_{n=1}^{\infty}$ by $i_n = u_n - 1$. By (5.5)-(5.6) and the mean value theorem,

$$|\pi_\beta(u) - \pi_{\beta'}(u)| = |g_{\mathbf{i}}(\beta^{-1}) - g_{\mathbf{i}}((\beta')^{-1})| = \left(\frac{1}{\beta} - \frac{1}{\beta'}\right)|g'_{\mathbf{i}}(x)| \leq \beta^{-2}(\beta' - \beta)|g'_{\mathbf{i}}(x)| \tag{5.7}$$

for some $x \in [1/\beta', 1/\beta] \subset [1/2, 1/\beta]$.

If $\beta > 1.5$, then $x \in [1/2, 2/3)$ and by Lemma 5.3, $|g'_{\mathbf{i}}(x)| \leq 2x \leq 2/\beta$, so by (5.7), $|\pi_\beta(u) - \pi_{\beta'}(u)| \leq 2\beta^{-3}(\beta' - \beta)$.

Next assume that $\sqrt{2} \leq \beta \leq 1.5$. Since $x \in [1/2, 1/\beta]$, either $x \in [1/2, 2/3)$ or $x \in [2/3, 3/4)$. If the first case occurs, then the argument in the last paragraph shows that $|\pi_\beta(u) - \pi_{\beta'}(u)| \leq 2\beta^{-3}(\beta' - \beta) \leq 3\beta^{-4}(\beta' - \beta)$. Else if $x \in [2/3, 3/4)$, then by

Lemma 5.3, $|g_{\mathbf{i}}'(x)| \leq 3x^2 \leq 3\beta^{-2}$, so by (5.7), $|\pi_\beta(u) - \pi_{\beta'}(u)| \leq 3\beta^{-4}(\beta' - \beta)$. This completes the proof of (i).

Next we prove (ii). Since $\mu_{\beta'} = m \circ \pi_{\beta'}^{-1}$ and $\mu_\beta = m \circ \pi_\beta^{-1}$, to prove (ii) it suffices to show that

$$\pi_{\beta'}^{-1}([c,d]) \subset \begin{cases} \pi_\beta^{-1}([c - 2\beta^{-3}(\beta' - \beta),\ d + 2\beta^{-3}(\beta' - \beta)]) & \text{if } \beta > 1.5, \\ \pi_\beta^{-1}([c - 3\beta^{-4}(\beta' - \beta),\ d + 3\beta^{-4}(\beta' - \beta)]) & \text{if } \beta \leq 1.5. \end{cases}$$

Clearly the above inclusion follows from (i) and we are done. □

Now we are ready to prove Proposition 5.2.

Proof of Proposition 5.2. Let $N \in \mathbb{N}$ and $\beta' \in [\beta, \beta + \delta]$ with $\beta' \leq 2$. Since $\mu_{\beta'}$ is supported on $[0,1]$ and has no atoms, it follows that $\mathscr{D}_{\beta,N}$ is a finite Borel partition of $\mathbb{R}$ with respect to $\mu_{\beta'}$. Applying Theorem 1.2(i) to the IFS $\{S_{1,\beta'},\ S_{2,\beta'}\}$ and the probability vector $(1/2, 1/2)$, we have

$$\dim(\mu_{\beta'}) \geq \frac{\log 2 - \sum_{[a,b] \in \mathscr{D}_{N,\beta}} f_2\left(\frac{1}{2}\mu_{\beta'}(S_{1,\beta'}^{-1}[a,b]),\ \frac{1}{2}\mu_{\beta'}(S_{2,\beta'}^{-1}[a,b])\right)}{\log \beta'}$$

By the above inequality and the increasing monotonicity of f_2, to prove (5.4) it suffices to show that for any $[a,b] \in \mathscr{D}_{\beta,N}$,

$$\mu_{\beta'}(S_{i,\beta'}^{-1}[a,b]) \leq \mu_\beta(S_{i,\beta}^{-1}[a - \epsilon, b + \epsilon]), \quad i = 1, 2, \tag{5.8}$$

where $\epsilon = \epsilon(\beta, \delta)$ is defined as in (5.3). To this end, set

$$\gamma = \begin{cases} 2\beta^{-3} & \text{if } \beta > 1.5, \\ 3\beta^{-4} & \text{if } \beta \leq 1.5. \end{cases}$$

Simply notice that

$$\begin{aligned} S_{1,\beta'}^{-1}[a,b] &= [\beta' a, \beta' b] \subset [\beta a, (\beta + \delta) b], \\ S_{2,\beta'}^{-1}[a,b] &= [\beta' a + 1 - \beta', \beta' b + 1 - \beta'] \subset [(\beta + \delta)(a - 1) + 1, \beta(b - 1) + 1], \end{aligned}$$

and

$$\begin{aligned} S_{1,\beta}^{-1}[a - \epsilon, b + \epsilon] &= [\beta a - \beta\epsilon, \beta b + \beta\epsilon], \\ S_{2,\beta}^{-1}[a - \epsilon, b + \epsilon] &= [\beta(a - \epsilon) + 1 - \beta, \beta(b + \epsilon) + 1 - \beta]. \end{aligned}$$

Since $\beta\epsilon = \delta(1 + \gamma)$ and $[a,b] \subset [0,1]$, we have

(5.9)

$$\begin{gathered} [\beta a - \gamma\delta, (\beta + \delta) b + \gamma\delta] \subset [\beta a - \beta\epsilon, \beta b + \beta\epsilon] \\ [(\beta + \delta)(a - 1) + 1 - \gamma\delta, \beta(b - 1) + 1 + \gamma\delta] \subset [\beta(a + \epsilon) + 1 - \beta, \beta(b + \epsilon) + 1 - \beta] \end{gathered}$$

Therefore for each $i \in \{1, 2\}$, the $\gamma\delta$-neighborhood of $S_{i,\beta'}^{-1}[a,b]$ is contained in $S_{i,\beta}^{-1}[a - \epsilon, b + \epsilon]$. Combining this fact with Lemma 5.4(ii) yields (5.8), and we are done. □

β	Lower bound on $\dim(\mu_\beta)$	N	Iteration times	δ	Time Consumed
1.42404000	0.990857395851368	5	28	2×10^{-5}	46.2962464
1.42406000	0.990863104536039	5	28	2×10^{-5}	46.2679773
1.42408000	0.990865644424569	5	28	2×10^{-5}	45.9752501
1.42410000	0.990869982602642	5	28	2×10^{-5}	46.6695526
1.42412000	0.990871200514580	5	28	2×10^{-5}	45.4384355
1.42414000	0.990876033055317	5	28	2×10^{-5}	45.3021147
1.42416000	0.990877731084033	5	28	2×10^{-5}	44.4827468
1.42418000	0.990883216572977	5	28	2×10^{-5}	44.6157998
1.42420000	0.990884812785932	5	28	2×10^{-5}	45.7283555
1.42422000	0.990887373414717	5	28	2×10^{-5}	46.1800343

TABLE 4. Lower bounds on $\dim(\mu_\beta)$ when β is near 1.42404; the unit for time consumption is in seconds.

β	Lower bound on $\dim(\mu_\beta)$	N	Iteration times	δ	Time Consumed
1.8392867549	0.980408601080113	13	40	10^{-10}	7.8168833
1.8392867550	0.980408591972940	13	40	10^{-10}	8.261404
1.8392867551	0.980408585976581	13	40	10^{-10}	8.1949713
1.8392867552	0.980408570326141	13	40	10^{-10}	8.5678704
1.8392867553	0.980408569070358	13	40	10^{-10}	8.3962353
1.8392867554	0.980408579672517	13	40	10^{-10}	8.3294095
1.8392867555	0.980408593496653	13	40	10^{-10}	8.524099
1.8392867556	0.980408601403820	13	40	10^{-10}	9.1053152
1.8392867557	0.980408603444209	13	40	10^{-10}	7.9313189
1.8392867558	0.980408612802920	13	40	10^{-10}	8.6475025

TABLE 5. Lower bounds on $\dim(\mu_\beta)$ for those β near $\beta_3 \approx 1.839286755214161$.

Proposition 5.2 provides a way to obtain a uniform lower bound on $\dim(\mu_\beta)$ when β varies in a given interval. For instance, let us take $\beta = 1.42404$, $\delta = 2 \times 10^{-5}$ and $N = 5$ in Proposition 5.2. A computation using (5.4) shows that

$$\dim(\mu_{\beta'}) \geq 0.990857395851368 \qquad \text{for all } \beta' \in [1.42404, 1.42406];$$

in which we used Algorithm 3.8 (with 28 iterations) to estimate $\mu_\beta(A)$ from above. Similarly, taking $\beta = 1.42406$, $\delta = 2 \times 10^{-5}$ and $N = 5$ in Proposition 5.2 gives

$$\dim(\mu_{\beta'}) \geq 0.990863104536039 \qquad \text{for all } \beta' \in [1.42406, 1.42408];$$

In Tables 4 and 5, we list our computational results for these (local) uniform lower bounds for those β near 1.42404 or near β_3, respectively.

Now are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 5.1(i), it suffices to show that $\dim(\mu_\beta) \geq 0.98040856$ for all $\beta \in [\sqrt{2}, 2]$, and $\dim(\mu_\beta) > \dim(\mu_{\beta_3})$ if

$$\beta \in [\sqrt{2}, 2] \backslash (\beta_3 - 10^{-8},\ \beta_3 + 10^{-8}).$$

Since $\dim(\mu_\beta) \geq 0.98041$ for $\beta \in [\sqrt{2}, 1.42404]$ (see Lemma 5.1(ii)), we only need to consider the parameters β in the interval $[1.42404, 2]$. To achieve our results, we further partition this interval into 132530 tiny intervals and use the algorithm developed in Proposition 5.2 to compute the (local) uniform lower bound of $\dim(\mu_\beta)$ on each of these tiny intervals. In Table 6, we give the precise information about our partition, as well as the choices of N, δ, and the iteration times (used for the estimations of μ_β using Algorithm 3.8) for each of these tiny intervals. For instance, the data listed in the second line in Table 6 mean that we partition the interval $[1.42404, 1.44]$ into sub-intervals of length 2×10^{-5}, and for each such subinterval we apply Proposition 5.2 to calculate the corresponding lower bound in which we take $N = 5$, $\delta = 2 \times 10^{-5}$ and 28 as the iteration times.

The full computational result on the (local) uniform lower bounds associated to these 132530 intervals is available at https://github.com/zfengg/DimEstimate. A graphic illustration of this result is given in Figure 1.

According to this computational result, the smallest lower bound that we obtained is 0.980408569070358, which is the uniform lower bound associated to the subinterval

$$[1.8392867553, 1.8392867554];$$

moreover, $\dim(\mu_\beta) > 0.9804094 > \dim(\mu_{\beta_3})$ for all

$$\beta \in [1.42404, 2] \backslash [1.8392867490, 1.8392867616].$$

That is enough to conclude Theorem 1.2. □

6. Other theoretical results on Bernoulli convolutions

For $n = 2, 3, \ldots$, let β_n be the largest root of the polynomial $x^n - x^{n-1} - x^{n-2} - \cdots - 1$. The first result of this section is the following.

Proposition 6.1. (i) *For every* $\beta \in (1, 2)$, $\dim(\mu_\beta) \geq \frac{\log 2}{\log \beta} \cdot \mu_\beta([0, \beta - 1])$.

(ii) *For each integer* $n \geq 2$,

$$\mu_{\beta_n}([0, \beta_n - 1]) = \frac{2^n - 2}{2^n - 1} \quad \text{and} \tag{6.1}$$

$$\mu_\beta([0, \beta - 1]) \geq \frac{2^n - 2}{2^n - 1} \quad \text{for } \beta \in [\beta_n,\ 2). \tag{6.2}$$

Consequently, $\dim(\mu_\beta) \geq \dfrac{2^n - 2}{2^n - 1} \cdot \dfrac{\log 2}{\log \beta_{n+1}}$ *for* $\beta \in [\beta_n, \beta_{n+1}]$, $n = 2, 3, \ldots$.

BetaStart	BetaEnd	N	Iteration times	BetaStep δ
1.42404	1.43998	5	28	2E-05
1.44	1.45998	5	28	2E-05
1.46	1.49998	5	28	2E-05
1.5	1.68999	5	30	1E-05
1.69	1.77999	6	30	1E-05
1.78	1.799999	7	40	1E-06
1.8	1.839199	7	40	1E-06
1.8392	1.8392599	7	40	1E-07
1.83926	1.83927399	7	40	1E-08
1.839274	1.8392863	10	40	1E-08
1.83928631	1.839286579	10	40	1E-09
1.83928658	1.8392869339	13	40	1E-10
1.839286934	1.839287249	10	40	1E-09
1.83928725	1.83929899	10	40	1E-08
1.839299	1.83930999	7	40	1E-08
1.83931	1.8399999	7	40	1E-07
1.84	1.849999	5	30	1E-06
1.85	1.99999	5	30	1E-05

TABLE 6. Partition of $[1.42404, 2]$ and the corresponding N, δ and iteration times.

Proof. We first prove (i). Fix $\beta \in (1, 2)$. Let

$$\mathscr{D} = \{[0, 1-\beta^{-1}), [1-\beta^{-1}, \beta^{-1}], (\beta^{-1}, 1]\},$$

which is a finite Borel partition of $[0, 1]$. Applying Theorem 1.2(i) to the IFS

$$\{S_{1,\beta}(x) = \beta^{-1}x,\ S_{2,\beta}(x) = \beta^{-1}x + 1 - \beta^{-1}\}$$

and the probability weight $(1/2, 1/2)$, we have

$$\dim(\mu_\beta) \geq \frac{\log 2 - \sum_{D\in\mathscr{D}} f_2\left(\frac{1}{2}\mu_\beta(S_{1,\beta}^{-1}D),\ \frac{1}{2}\mu_\beta(S_{2,\beta}^{-1}D)\right)}{\log \beta}. \tag{6.3}$$

It is easily checked that $f_2(x, y) = 0$ if one of x and y equals 0, and $f_2(x, x) = 2x \log 2$. Meanwhile if $D = [0, 1-\beta^{-1})$ or $(\beta^{-1}, 1]$, then one of $S_{1,\beta}^{-1}D$ and $S_{2,\beta}^{-1}D$ has no

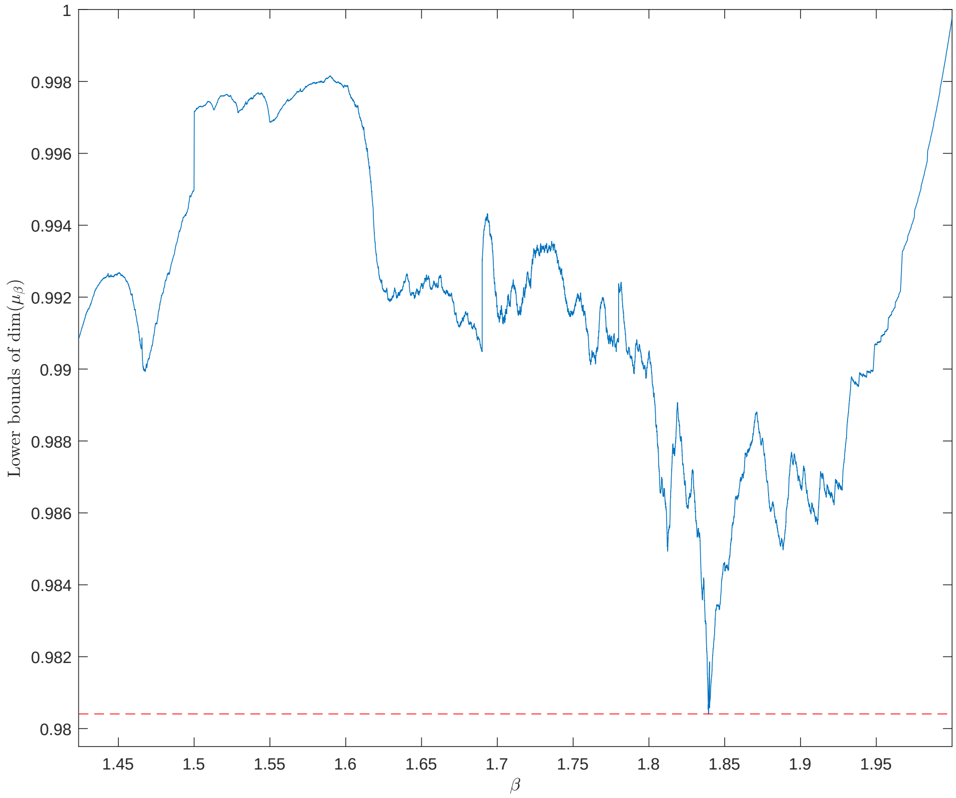

FIGURE 1. A graphic illustration of our computational result on the (local) uniform lower bounds on $\dim(\mu_\beta)$.

intersections with $[0,1]$, so has zero μ_β measure. It follows that

$$\begin{aligned}
&\sum_{D\in\mathscr{D}} f_2\left(\frac{1}{2}\mu_\beta(S_{1,\beta}^{-1}D),\ \frac{1}{2}\mu_\beta(S_{2,\beta}^{-1}D)\right)\\
&\qquad = f_2\left(\frac{1}{2}\mu_\beta(S_{1,\beta}^{-1}[1-\beta^{-1},\beta^{-1}]),\ \frac{1}{2}\mu_\beta(S_{2,\beta}^{-1}[1-\beta^{-1},\beta^{-1}])\right)\\
&\qquad = f_2\left(\frac{1}{2}\mu_\beta([\beta-1,1]),\ \frac{1}{2}\mu_\beta([0,2-\beta])\right)\\
&\qquad = \mu_\beta([\beta-1,1])\log 2,
\end{aligned}$$

where in the last equality we use the property that $\mu_\beta([\beta-1,1]) = \mu_\beta([0,2-\beta])$, which follows from the symmetry of μ_β (i.e. $\mu_\beta([0,x]) = \mu_\beta([1-x,1])$ for all $x \in [0,1]$). Plugging the above equality into (6.3) yields the desired inequality in (i).

Next we prove (6.1). Fix an integer $n \geq 2$ and write $\rho := \beta_n^{-1}$. We claim that

$$\mu_{\beta_n}([0, 1-\rho]) = \frac{1}{2}\mu_{\beta_n}([0, \rho^{-1}-1]) = \frac{1}{2}\mu_{\beta_n}([0, 1-\rho^n]), \tag{6.4}$$

$$\mu_{\beta_n}([0, 1-\rho^k]) = \frac{1}{2} + \frac{1}{2}\mu_{\beta_n}([0, 1-\rho^{k-1}]), \quad k = 2, \ldots, n, \quad \text{and} \tag{6.5}$$

To see (6.4), by the self-similarity of μ_{β_n}, we have

$$\mu_{\beta_n}([0, 1-\rho]) = \frac{1}{2}\mu_{\beta_n}([0, \rho^{-1}-1]) + \frac{1}{2}\mu_{\beta_n}([1-\rho^{-1}, 0]) = \frac{1}{2}\mu_{\beta_n}([0, \rho^{-1}-1]).$$

This proves (6.4), since $\rho^{-1} - 1 = 1 - \rho^n$. Similarly for $2 \leq k \leq n$,

$$\begin{aligned}\mu_{\beta_n}([0, 1-\rho^k]) &= \frac{1}{2}\mu_{\beta_n}([0, \rho^{-1}-\rho^{k-1}]) + \frac{1}{2}\mu_{\beta_n}([1-\rho^{-1}, 1-\rho^{k-1}]) \\ &= \frac{1}{2} + \frac{1}{2}\mu_{\beta_n}([0, 1-\rho^{k-1}]),\end{aligned}$$

where in the second equality we use the fact that

$$\rho^{-1} - \rho^{k-1} = (1 + \rho + \cdots + \rho^{n-1}) - \rho^{k-1} \geq 1.$$

This proves (6.5). Solving the linear equations in (6.4)-(6.5) gives

$$\mu_{\beta_n}([0, \rho^{-1}-1]) = \mu_{\beta_n}([0, 1-\rho^n]) = \frac{2^n-2}{2^n-1},$$

which proves (6.1).

Finally we prove (6.2). Fix an integer $n \geq 2$ and $\beta \in [\beta_n, 2)$. We first show that

$$\beta(\beta^{n-1} - \beta^{n-2} - \cdots - 1) \geq 1, \tag{6.6}$$

and

$$\beta(\beta-1)(\beta^i - \beta^{i-1} - \cdots - 1) \geq 1 \text{ for } i = 1, \ldots, n-2, \text{ provided that } n \geq 3, \tag{6.7}$$

To prove the above inequalities, notice that

$$\beta \geq \beta_n = 1 + \beta_n^{-1} + \cdots + \beta_n^{-(n-1)} \geq 1 + \beta^{-1} + \cdots + \beta^{-(n-1)}.$$

Hence

$$\beta^{n-1} - \beta^{n-2} - \cdots - 1 \geq \beta^{n-2}(1 + \beta^{-1} + \cdots + \beta^{-(n-1)}) - \beta^{n-2} - \cdots - 1 = \beta^{-1},$$

from which (6.6) follows. Moreover, for $i = 1, \ldots, n-2$ (provided that $n \geq 3$),

$$\begin{aligned}\beta^i - \beta^{i-1} - \ldots - 1 &\geq \beta^{i-1}(1 + \beta^{-1} + \cdots + \beta^{-(n-1)}) - \beta^{i-1} - \cdots - 1 \\ &= \beta^{-1} + \cdots + \beta^{i-n} \\ &\geq \beta^{-1} + \beta^{-2},\end{aligned}$$

so

$$\beta(\beta-1)(\beta^i - \beta^{i-1} - \cdots - 1) \geq \beta(\beta-1)(\beta^{-1} + \beta^{-2}) = \beta - \beta^{-1} \geq 1.$$

This proves (6.7).

By the self-similarity of μ_β, we have

$$\begin{aligned}\mu_\beta([0,\beta-1]) &= \frac{1}{2}\mu_\beta([0,\beta^2-\beta]) + \frac{1}{2}\mu_\beta([0,(\beta-1)(\beta-1)]) \\ &= \frac{1}{2} + \frac{1}{2}\mu_\beta([0,(\beta-1)(\beta-1)]),\end{aligned} \tag{6.8}$$

where in the second equality we use the fact that $\beta^2-\beta\geq 1$. Similarly we have

$$\begin{aligned}&\mu_\beta([0,(\beta-1)(\beta^{n-1}-\beta^{n-2}-\cdots-1)]) \\ &\geq \frac{1}{2}\mu_\beta([0,\beta(\beta-1)(\beta^{n-1}-\beta^{n-2}-\cdots-1)]) \\ &\geq \frac{1}{2}\mu_\beta([0,\beta-1]) \qquad \text{(by (6.6))}.\end{aligned} \tag{6.9}$$

Moreover, for $i=1,\ldots,n-2$ (provided $n\geq 3$), we have

$$\begin{aligned}&\mu_\beta([0,(\beta-1)(\beta^i-\beta^{i-1}-\cdots-1)]) \\ &= \frac{1}{2}\mu_\beta([0,\beta(\beta-1)(\beta^i-\beta^{i-1}-\cdots-1)]) + \frac{1}{2}\mu_\beta([0,(\beta-1)(\beta^{i+1}-\beta^i-\cdots-1)]) \\ &= \frac{1}{2} + \frac{1}{2}\mu_\beta([0,(\beta-1)(\beta^{i+1}-\beta^i-\cdots-1)]) \qquad \text{(by (6.7))}.\end{aligned} \tag{6.10}$$

To complete the proof of (6.2), we consider the cases $n=2$ and $n\geq 3$ separately. First assume that $n=2$. Then by (6.8)-(6.9), we have

$$\mu_\beta([0,\beta-1]) = \frac{1}{2}+\frac{1}{2}\mu_\beta([0,(\beta-1)^2]) \quad \text{and} \quad \mu_\beta([0,(\beta-1)^2]) \geq \frac{1}{2}\mu_\beta([0,\beta-1]),$$

from which we deduce that $\mu_\beta([0,\beta-1])\geq 2/3$. This proves (6.2) in the case that $n=2$.

In what follows we assume $n\geq 3$. Set $x=\mu_\beta([0,\beta-1])$ and

$$y_i = \mu_\beta([0,(\beta-1)(\beta^i-\beta^{i-1}-\cdots-1)]) \quad \text{for } i=1,\ldots,n-1.$$

By (6.8) and (6.10), we have

$$x=\frac{1}{2}+\frac{1}{2}y_1 \quad \text{and} \quad y_i=\frac{1}{2}+\frac{1}{2}y_{i+1}, \quad i=1,\ldots,n-2,$$

from which we deduce that $y_{n-1}=2^{n-1}x-(2^{n-1}-1)$. Meanwhile by (6.9) we have $y_{n-1}\geq x/2$. Hence $2^{n-1}x-(2^{n-1}-1)\geq x/2$, from which we obtain

$$\mu_\beta([0,\beta-1]) = x \geq \frac{2^n-2}{2^n-1}.$$

This completes the proof of (6.2) and we are done. $\square$

Remark 6.2. It is known that $\beta_n = 2 - 2^{-n} + O(\frac{n}{4^n})$ (see [12, Lemma 3.3]). Applying this fact and Proposition 6.1, it is easy to show that there exists a constant $c > 0$ such that

$$1 - \dim \mu_\beta \leq c(2 - \beta) \text{ for all } \beta \in (1, 2).$$

This plays a complement to the following inequality obtained in [23, Theorem 3]:

$$1 - \dim \mu_\beta \leq c(\beta - 1) \text{ for some } c > 0 \text{ and all } \beta \in (1, 2).$$

Meanwhile, according to the theoretic formula of $\dim(\mu_{\beta_n})$ (see [5, 8, 14]), one can check that

$$\dim(\mu_{\beta_n}) = 1 - \left(1 - \frac{1}{\log 4}\right) 2^{-n} + O\left(\frac{n}{4^n}\right),$$

which implies that

$$\lim_{n\to\infty} \frac{1 - \dim(\mu_{\beta_n})}{2 - \beta_n} = 1 - \frac{1}{\log 4}.$$

In the remaining part of this section, we prove the following.

Lemma 6.3. *There exists* $\beta_* \in (\sqrt{2}, 2)$ *such that*

$$\dim_H(\mu_{\beta_*}) = \inf_{\beta\in(1,2)} \dim(\mu_\beta).$$

Proof. Notice that $\dim(\mu_\beta) = 1$ if $\beta = 2$. Using the inequality $\dim(\mu_\beta) \geq \dim(\mu_{\beta^2})$ (cf. Lemma 5.1(i)) yields that $\dim(\mu_{\sqrt{2}}) = 1$.

For each $\beta \in (1, \sqrt{2})$, there exists a positive integer k such that $\beta^k \in [\sqrt{2}, 2]$. By Lemma 5.1, $\dim(\mu_\beta) \geq \dim(\mu_{\beta^k})$. It follows that

$$\inf_{\beta\in(1,2)} \dim(\mu_\beta) = \inf_{\beta\in[\sqrt{2},2]} \dim(\mu_\beta) = \inf_{\beta\in(\sqrt{2},2)} \dim(\mu_\beta).$$

It is known that the mapping $\beta \mapsto \dim(\mu_\beta)$ is lower semi-continuous on $(1, 2]$ (see e.g. [20, Theorem 1.8]). Hence there exists $\beta_* \in (\sqrt{2}, 2)$ so that

$$\dim_H(\mu_{\beta_*}) = \inf_{\beta\in[\sqrt{2},2]} \dim(\mu_\beta) = \inf_{\beta\in(1,2)} \dim(\mu_\beta).$$

This proves the lemma. □

7. Upper bound estimate for the dimension of Bernoulli convolutions associated with Pisot numbers

In this section we prove Theorem 1.4 and give our computational results on the upper and lower bounds on $\dim(\mu_\beta)$ for some examples of Pisot numbers β of degree 3 or 4.

We begin with the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $P(q)$, $q>0$, be defined as in (1.7). It is proved in [10, Theorems 3.3-3.4] that P is differentiable on $(0,\infty)$ and for each $q>0$, there is a unique σ-invariant measure η_q on Σ satisfying the following Gibbs property

$$\eta_q([i_1\cdots i_n])\approx e^{-nP(q)}\|A_{i_1}\cdots A_{i_n}\|^q \quad \text{for } n\in\mathbb{N} \text{ and } i_1\cdots i_n\in\{1,\ldots,k\}^n,$$

where $\|\cdot\|$ stands for the standard matrix norm. Furthermore by [7, Theorems 1.1, 1.2 and 3.1], P satisfies the following variational relation

$$P(q)=h_{\eta_q}(\sigma)+q\lim_{n\to\infty}\frac{1}{n}\int\log\|A_{x|n}\|\,d\eta_q(x), \qquad q>0$$

and the derivative formula

$$P'(q)=\lim_{n\to\infty}\frac{1}{n}\int\log\|A_{x|n}\|\,d\eta_q(x), \qquad q>0,$$

where $A_{x|n}:=A_{x_1}\cdots A_{x_n}$ for $x=(x_n)_{n=1}^\infty\in\Sigma$. Combining the above two equalities yields that $P(q)=h_{\eta_q}(\sigma)+qP'(q)$. Taking $q=1$ and using the fact that $P(1)=0$ we obtain $P'(1)=-h_{\eta_1}(\sigma)$. By the definition of η (see (1.8)), η satisfies the Gibbs property $\eta([i_1\cdots i_n])\approx\|A_{i_1}\cdots A_{i_n}\|$, so $\eta=\eta_1$. Thus $P'(1)=-h_\eta(\sigma)$. Combining it with (1.6) yields $\dim(\mu_\beta)=h_\eta(\sigma)/(\log\beta)$.

To complete our proof, let $\mathcal{P}=\{[i]:\ i=1,\ldots,k\}$ be the partition of $\Sigma:=\{1,\ldots,k\}^{\mathbb{N}}$ consisting of the first order cylinders in Σ. Then

$$\begin{aligned}
u_n &:= \sum_{I\in\{1,\ldots,k\}^{n+1}}\varphi\left(\eta([I])\right)-\sum_{J\in\{1,\ldots,k\}^n}\varphi\left(\eta([J])\right)\\
&= H_\eta\left(\mathcal{P}\vee\cdots\vee\sigma^{-n}\mathcal{P}\right)-H_\eta\left(\mathcal{P}\vee\cdots\vee\sigma^{-(n-1)}\mathcal{P}\right)\\
&= H_\eta\left(\mathcal{P}\vee\cdots\vee\sigma^{-n}\mathcal{P}\right)-H_\eta\left(\sigma^{-1}\mathcal{P}\vee\cdots\vee\sigma^{-n}\mathcal{P}\right)\\
&= H_\eta\left(\mathcal{P}|\left(\sigma^{-1}\mathcal{P}\vee\cdots\vee\sigma^{-n}\mathcal{P}\right)\right),
\end{aligned}$$

where we used [30, Theorem 4.3(ii)] in the last equality. By [30, Theorem 4.3(iii)], the sequence (u_n) is decreasing. Now by definition,

$$\begin{aligned}
h_\eta(\sigma) &= \lim_{n\to\infty}\frac{1}{n}H_\eta\left(\mathcal{P}\vee\cdots\vee\sigma^{-(n-1)}\mathcal{P}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{|I|=n}\varphi\left(\eta([I])\right)\\
&= \lim_{n\to\infty}\frac{1}{n}(u_n+u_{n-1}+\cdots+u_1)
\end{aligned}$$

Since (u_n) is decreasing, it follows that $h_\eta(\sigma)=\lim_{n\to\infty}u_n=\inf_n u_n$. This completes the proof. □

Level N	Low bound	Time consumed	Level n	Upper bound	Time consumed
10	0.999446403056440	0.054791699	6	0.999553838975762	0.466657771
11	0.999485559701407	0.019805691	7	0.999551745301729	0.077114112
12	0.999511565006248	0.018707125	8	0.999549324828679	0.098623572
13	0.999524092865024	0.031473356	9	0.999548108265720	0.099398655
14	0.999533483697397	0.027698097	10	0.999546974504613	0.122857504
15	0.999537083624124	0.035850279	11	0.999546321331208	0.138546403
16	0.999539945814714	0.054405927	12	0.999545837403620	0.198051248
17	0.999541706067639	0.063491041	13	0.999545467625853	0.247629358
18	0.999542557004638	0.071829385	14	0.999545258823762	0.312467417
19	0.999543459170884	1.607735407	15	0.999545072687759	0.584398307
20	0.999543735484899	0.186936405	16	0.999544971532786	0.618315478
21	0.999544108747012	0.208533991	17	0.999544888243190	1.015030255
22	0.999544301645519	0.243305524	18	0.999544831886938	1.195914636
23	0.999544402854973	0.355187983	19	0.999544797950427	1.611427142
24	0.999544527192754	0.563398845	20	0.999544767910898	2.332147467
25	0.999544565385905	3.173328553	21	0.999544751961036	3.621352815
26	0.999544619835200	1.163999234	22	0.999544738388558	5.046995913
27	0.999544645262069	1.653751317	23	0.999544729649307	8.976078225
28	0.999544663526862	2.458708739	24	0.999544723974222	10.85274788
29	0.999544681239203	3.52355423	25	0.999544719277577	15.76524517
30	0.999544687025939	5.013234467	26	0.999544716737165	22.61266586
31	0.999544695999151	7.36229044	27	0.999544714509664	33.20685747
32	0.999544699587091	10.73880172	28	0.999544713164084	48.45996338
33	0.999544702739358	15.77259785	29	0.999544712224285	71.26915581
34	0.999544705355331	23.01813448	30	0.999544711488671	104.4124192
35	0.999544706362411	34.01411608	31	0.999544711078597	152.6973032

TABLE 7. Lower and upper bounds on $\dim(\mu_\beta)$ where $\beta \approx 1.465571232$ is the largest root of $x^3 - x^2 - 1$; the unit for time consumption is in seconds.

In the remaining part of this section we give some computational results on the upper and lower bounds on $\dim(\mu_\beta)$ for some examples of Pisot numbers β of degree 3 or 4.

We begin with the illustration of the computations for the example in which $\beta \approx 1.465571232$ is the largest root of the polynomial $x^3 - x^2 - 1 = 0$. We use the inequality (1.2) in Theorem 1.1 to give a finite sequence of low bounds on $\dim(\mu_\beta)$, where $\mathscr{D} = \mathscr{D}_{N,\beta}$ is the partition of $[0,1]$ generated by the points in the set given in (5.2), whilst we use the inequality (1.9) in Theorem 1.4 to give a finite sequence of upper bounds on $\dim(\mu_\beta)$; in this example, there are 46 constructed matrices A_i's of dimension $d = 346$. In Table 7 we list these lower and upper bounds.

In a similar way we compute the lower and upper bounds on $\dim(\mu_\beta)$ for 4 other Pisot numbers of degree 3 or 4. In Table 8, we list the corresponding computational results briefly.

Polynomial	β	lower bound on $\dim(\mu_\beta)$	upper bound on $\dim(\mu_\beta)$
x^3-x^2-1	1.465571232	0.999544706362411	0.999544711078597
x^3-x-1	1.324717957	0.999995036655607	0.999995037372877
x^3-2x^2+x-1	1.754877666	0.994020046394375	0.994020065372927
x^4-2x^3+x-1	1.866760399	0.991391363780141	0.991401387766015
$x^4-x^3-2x^2+1$	1.905166167	0.989449155226028	0.989601151164740

TABLE 8. Lower and upper bounds on $\dim(\mu_\beta)$ for some Pisot numbers β of degree 3 or 4.

8. Final remarks

In this section, we give several final remarks.

Remark 8.1. One could obtain further sharper uniform lower bounds on the dimension of Bernoulli convolutions if he/she partitions the interval $[\beta_3-10^{-8},\beta_3+10^{-8}]$ into subintervals of length much smaller than 10^{-10} and manages to compute the (local) unform lower bounds of $\dim\mu_\beta$ associated to these subintervals (in which taking $N>13$ and an iteration time $L>40$).

Remark 8.2. The method of estimating projection entropies can be also used to find lower bounds for the dimension of certain self-affine measures. To be more precise, let μ be the self-affine measure associated with an affine IFS $\{S_i(x)=A_ix+a_i\}_{i=1}^{\ell}$ on $\mathbb{R}^d$ and a probability vector $(p_1,\ldots,p_\ell)$, where A_i are contracting invertible $d\times d$ matrices. Suppose that these matrices are all diagonal, then $\dim(\mu)$ can be expressed as the linear combinations of projection entropies associated with several coding maps (see [9, Theorems 2.11-2.12]), hence we can provide lower bounds for $\dim(\mu)$ by estimating these projection entropies. Below we give a concrete example.

Example 8.3. Let μ be the self-affine measure associated with an affine IFS $\{S_1,S_2\}$ on $\mathbb{R}^2$ and the probability vector $(1/2,1/2)$, where

$$S_1(x,y)=\left(\frac{x}{\alpha},\ \frac{y}{\beta}\right),\quad S_2(x,y)=\left(\frac{x}{\alpha}+1-\frac{1}{\alpha},\ \frac{y}{\beta}+1-\frac{1}{\beta}\right),$$

and α,β are two parameters with $1<\alpha<\beta<2$. Let $m=\prod_{n=1}^{\infty}\{1/2,1/2\}$, and let π, π_1 denote the canonical coding maps associated with the IFSs $\{S_1,S_2\}$ and $\{\alpha^{-1}x,\alpha^{-1}x+1-\alpha^{-1}\}$, respectively. Then by [9, Theorem 2.11],

$$\begin{aligned}\dim(\mu)&=\left(\frac{1}{\log\alpha}-\frac{1}{\log\beta}\right)h_{\pi_1}(m)+\frac{1}{\log\beta}h_\pi(m)\\&=\left(\frac{1}{\log\alpha}-\frac{1}{\log\beta}\right)\big(\log 2-H_m(\mathcal{P}|\pi_1^{-1}\mathcal{B}(\mathbb{R})\big)+\frac{\log 2-H_m\left(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^2)\right)}{\log\beta},\end{aligned}$$

(α, β)	Upper bound on $H_m(\mathcal{P}\|\pi_1^{-1}\mathcal{B}(\mathbb{R}))$	Upper bound on $H_m(\mathcal{P}\|\pi^{-1}\mathcal{B}(\mathbb{R}^2))$	Lower bound on $\dim(\mu)$
$(1.2, 1.4)$	0.594519457635335	0.381337271355742	1.17453512577913
$(1.3, 1.7)$	0.443677110679849	0.011983272405036	1.76440615371483
$(1.4, 1.8)$	0.358042443198116	0.000020606397791	1.60503743978513
$(1.5, 1.6)$	0.287856436058623	0.010339956295331	1.59002600229069
$(1.5, 1.7)$	0.287856436058623	0.000000000000000	1.54205227015933
$(1.7, 1.71)$	0.165391636766638	0.105399412288277	1.10640907763501
$(1.8, 1.9)$	0.064653016798699	0.035399973460099	1.03287878328719

TABLE 9. Lower bounds on the dimension of μ in Example 8.3.

where $\mathcal{P} = \{[1], [2]\}$ is the natural partition of $\{1, 2\}^{\mathbb{N}}$. For each given pair (α, β), we can apply (3.3) to numerically estimate the conditional entropies $H_m(\mathcal{P}|\pi_1^{-1}\mathcal{B}(\mathbb{R}))$ and $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^2))$ from above, which will lead to a lower bound on $\dim \mu$. In Table 9, we present our computations results for some chosen parameters (α, β).

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