Suggested Solution to HW3[1]

MATH3270A

Fall Semester 2017-18

1. For a vector $\vec{y}^{(k)}$ with n components, we adopt the notation $\vec{y}^{(k)}(t) = \left(y_1^{(k)}, y_2^{(k)}, \cdots, y_n^{(k)}\right)^T$. The fundamental matrix is given by

$$\mathbb{F}(t) := \begin{pmatrix} y_1^{(1)} & y_1^{(2)} & y_1^{(3)} \\ y_2^{(1)} & y_2^{(2)} & y_2^{(3)} \\ y_3^{(1)} & y_3^{(2)} & y_3^{(3)} \end{pmatrix},$$

and this matrix solves

$$\mathbb{F}'(t) = \mathbb{P}(t)\mathbb{G}(t) = \sum_{k=1}^{3} \begin{pmatrix} P_{1k}y_k^{(1)} & P_{1k}y_k^{(2)} & P_{1k}y_k^{(3)} \\ P_{2k}y_k^{(1)} & P_{2k}y_k^{(2)} & P_{2k}y_k^{(3)} \\ P_{3k}y_k^{(1)} & P_{3k}y_k^{(2)} & P_{3k}y_k^{(3)} \end{pmatrix}. \quad (1)$$

In addition the Wronskian $W(t) = \det(\mathbb{G}(t))$ and then

$$\frac{d}{dt}W(t) = \begin{vmatrix} \frac{d}{dt}y_1^{(1)} & \frac{d}{dt}y_1^{(2)} & \frac{d}{dt}y_1^{(3)} \\ y_2^{(1)} & y_2^{(2)} & y_2^{(3)} \\ y_3^{(1)} & y_3^{(2)} & y_3^{(3)} \end{vmatrix} + \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & y_1^{(3)} \\ \frac{d}{dt}y_2^{(1)} & \frac{d}{dt}y_2^{(2)} & \frac{d}{dt}y_2^{(3)} \\ y_3^{(1)} & y_3^{(2)} & y_3^{(3)} \end{vmatrix} + \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & y_1^{(3)} \\ y_2^{(1)} & y_2^{(2)} & y_2^{(3)} \\ \frac{d}{dt}y_3^{(1)} & \frac{d}{dt}y_3^{(2)} & \frac{d}{dt}y_3^{(3)} \end{vmatrix}$$

$$=: I_1 + I_2 + I_3.$$

By the system (1) satisfied by $\mathbb{F}(t)$,

$$I_1 = \sum_{k=1}^{3} \begin{vmatrix} P_{1k}y_k^{(1)} & P_{1k}y_k^{(2)} & P_{1k}y_k^{(3)} \\ y_2^{(1)} & y_2^{(2)} & y_2^{(3)} \\ y_3^{(1)} & y_3^{(2)} & y_3^{(3)} \end{vmatrix} = P_{11}W(t) + 0 + 0 = P_{11}W(t).$$

Similarly one has

$$I_2 = P_{22}W(t), \quad I_3 = P_{33}W(t).$$

Therefore

$$\frac{d}{dt}W(t) = (P_{11} + P_{22} + P_{33})W(t). \quad \textbf{[5pts]}$$

2. (a) The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = 0 \iff \lambda = \pm i.$$

All these roots are repeated, hence the fundamental set of solutions is

$$S := \{\cos t, \sin t, t\cos t, t\sin t\}. \quad \textbf{[1pt]}$$

Now we find a particular solution $Y(t) = Y_1(t) + Y_2(t)$ with

$$\begin{cases} Y_1^{(4)} + 2Y_1'' + Y_1 = 8; \\ Y_2^{(4)} + 2Y_2'' + Y_2 = \sin 2t \end{cases} \quad (2)$$

[1] By Yuan Chen

Obviously, $Y_1 = 8$ is a solution **[1pt]**. Since $\sin 2t \notin S$, take $Y_2 = B_1 \cos 2t + B_2 \sin 2t$, then

$$Y_2'' = -4B_1 \cos 2t - 4B_2 \sin 2t; \quad Y_2^{(4)} = 16B_1 \cos 2t + 16 \sin 2t.$$

Then

$$\sin 2t = Y_2^{(4)} + 2Y_2'' + Y_2 = 9B_1 \cos 2t + 9B_2 \sin 2t \quad \text{with} \quad B_1 = 0, B_2 = \frac{1}{9}. \textbf{[1pt]}$$

As a consequence, the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t + \frac{1}{9} \sin 2t + 8$$

for some constants c_1, c_2, c_3, c_4.

(b) The corresponding characteristic equation is

$$\lambda^3 + \lambda^2 + \lambda + 1 = (\lambda + 1)(\lambda^2 + 1) = 0 \quad \Longleftrightarrow \quad \lambda = -1, \pm i.$$

Hence the fundamental set of solutions to the homogeneous equation is

$$S := \{e^{-t}, \cos t, \sin t\}. \qquad \text{[1pt]}$$

A particular solution $Y(t)$ is given by $Y(t) = Y_1(t) + Y_2(t)$ with

$$\begin{cases} Y_1''' + Y_1'' + Y_1' + Y_1 = 2e^{-t}; \\ Y_2''' + Y_2'' + Y_2' + Y_2 = 5t^2. \end{cases}$$

Firstly, we consider the first equation. Since $e^{-t} \in S$, take $Y_1(t) = Ate^{-t}$ **[0.5pt]**, then

$$Y_1' = A(1-t)e^{-t}, \quad Y_1'' = A(t-2)e^{-t}, \quad Y_1''' = A(3-t)e^{-t}.$$

And for $A = 1$, $Y_1 = te^t$ **[0.5pt]**,

$$Y_1''' + Y_1'' + Y_1' + Y_1 = A\big[t + 1 - t + t - 2 + 3 - t\big]e^{-t} = 2Ae^{-t} = 2e^{-t}.$$

Secondly, take $Y_2(t) = B_1 t^2 + B_2 t + B_3$ **[0.5pt]**, then

$$Y_2' = 2B_1 t + B_2, \quad Y_2'' = 2B_1, \quad Y_2''' = 0,$$

and then

$$Y_2''' + Y_2'' + Y_2' + Y_2 = 2B_1 + B_2 + B_3 + (2B_1 + B_2)t + B_1 t^2 = 5t^2.$$

It is easy to get $B_1 = 5, B_2 = -10, B_3 = 0$, and $Y_2 = 5t^2 - 10t$ **[0.5pt]**. Therefore the general solution is

$$y(t) = c_1 e^{-t} + c_2 \cos t + c_3 \sin t + te^{-t} + 5t^2 - 10t$$

for some constants c_1, c_2, c_3.

(c) The characteristic equation is

$$\lambda^3 + \lambda = \lambda(\lambda^2 + 1) = 0 \quad \Longleftrightarrow \quad \lambda = 0, \pm i.$$

This implies the fundamental solutions to the homogeneous equation are

$$y_1 = 1, \quad y_2 = \cos t, \quad y_3 = \sin t. \quad \textbf{[1pt]}$$

Now we proceed to find a particular solution in the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t),$$

where u_1, u_2, u_3 solves

$$\begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = (\tan t + 5\sec t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \textbf{[1pt]}$$

It is easy to calculate

$$\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1.$$

By Cramer's rule,

$$u_1' = (\tan t + 5\sec t) \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = \tan t + 5\sec t,$$

and then

$$u_1(t) = -\ln|\cos t| + 5\ln|\sec t + \tan t|.$$

Still by Cramer's rule,

$$u_2'(t) = (\tan t + 5\sec t) \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -(\tan t + 5\sec t)\cos t = -\sin t - 5.$$

Integrating yields

$$u_2(t) = \cos t - 5t.$$

Similar argument gives

$$u_3'(t) = (\tan t + 5\sec t) \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sec t + \cos t - 5\tan t,$$

Integrating leads

$$u_3 = -\ln|\sec t + \tan t| + \sin t + 5\ln|\cos t|.$$

As a consequence of above arguments, the general solution is

$$\begin{aligned} y(t) =& c_1 - \ln|\cos t| + 5\ln|\sec t + \tan t| + \big(c_2 + \cos t - 5t\big)\cos t \\ &+ \big(c_3 - \ln|\sec t + \tan t| + \sin t + 5\ln|\cos t|\big)\sin t. \quad \textbf{[1pt]} \end{aligned}$$

for some constants c_1, c_2, c_3.

(d) The only thing need to do is to find a particular solution. One may consider a solution in the form

$$Y(t) = tu_1(t) + t^2u_2(t) + \frac{1}{t}u_3(t),$$

with

$$\begin{pmatrix} t & t^2 & 1/t \\ 1 & 2t & -1/t^2 \\ 0 & 2 & 2/t^3 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = 2t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \textbf{[1pt]}$$

Firstly,

$$\begin{vmatrix} t & t^2 & 1/t \\ 1 & 2t & -1/t^2 \\ 0 & 2 & 2/t^3 \end{vmatrix} = \frac{6}{t}.$$

Then Cramer's rule and direct calculations show

$$u_1' = -t^2, \quad u_2'(t) = \frac{2}{3}t, \quad u_3'(t) = \frac{1}{3}t^4. \quad \textbf{[1pt]}$$

Integrating yields

$$u_1 = -\frac{1}{3}t^3, \quad u_2 = \frac{1}{3}t^2, \quad u_3 = \frac{1}{15}t^5.$$

Therefore the particular solution is

$$Y(t) = -\frac{1}{3}t^4 + \frac{1}{3}t^4 + \frac{1}{15}t^4 = \frac{1}{15}t^4. \quad \textbf{[1pt]}$$

Consequently, the general solution is in the form

$$y(t) = c_1 t + c_2 t^2 + \frac{c_3}{t} + \frac{1}{15}t^4.$$

3. (a) Let $\vec{Y}(t) = (Y_1, Y_2, Y_3)(t)$. $\mathbb{A}$ is a Jordan matrix, the corresponding linear system is equivalent to

$$\begin{cases} Y_1' = 2Y_1 + Y_2, \\ Y_2' = 2Y_2 + Y_3, \\ Y_3' = 2Y_3. \end{cases} \iff \begin{cases} (e^{-2t}Y_1)' = e^{-2t}Y_2, \\ (e^{-2t}Y_2)' = e^{-2t}Y_3, \\ (e^{-2t}Y_3)' = 0. \end{cases}$$

Solving the third equation first leads to

$$Y_3 = c_1 e^{2t}$$

for an arbitrary constant c_1. Now solving the second equation gives

$$Y_2(t) = (c_1 t + c_2)e^{2t}.$$

And lastly the first one,

$$Y_1(t) = \left(\frac{c_1}{2}t^2 + c_2 t + c_3\right)e^{2t}.$$

As a consequence, the general solution is

$$\vec{Y}(t) = \begin{pmatrix} c_1 t^2/2 + c_2 t + c_3 \\ c_1 t + c_2 \\ c_1 \end{pmatrix} e^{2t} \quad \textbf{[3pts]}$$

for some constants c_1, c_2, c_3.

(Alternative) Obviously, the only eigenvalue is $\lambda = 2$ associated with one eigenvector $\vec{\xi} = (1, 0, 0)^T$. This gives one solution

$$\vec{x}^{(1)}(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad \textbf{[1pt]}$$

To find a second independent solution, we first find a generalized eigenvector $\vec{\eta}$ solving

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = (\mathbb{A} - 2\mathbb{I})\vec{\eta} = \vec{\xi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

which is equivalent to $\eta_2 = 1, \eta_3 = 0$. Hence one may take $\vec{\eta} = (0,1,0)^T$, and the second independent solution is given by

$$\vec{x}^{(2)}(t) = e^{2t}\big(t\vec{\xi} + \vec{\eta}\big) = e^{2t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}. \quad \textbf{[1pt]}$$

Lastly, we are going to find the third independent solution.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = (\mathbb{A} - 2\mathbb{I})\vec{\theta} = \vec{\eta} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

which yields $\theta_2 = 0, \theta_3 = 1$. Thus one may take $\vec{\theta} = (0,0,1)^T$. And the third independent solution is

$$\vec{x}^{(3)}(t) = e^{2t}\left(\frac{t^2}{2}\vec{\xi} + t\vec{\eta} + \vec{\theta}\right) = e^{2t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix}. \quad \textbf{[1pt]}$$

Consequently, the general solution is

$$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix}$$

for some constants c_1, c_2, c_3.

(b) The eigenvalues of $\mathbb{A}$ are given by

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\big[(1-\lambda)^2 + 4\big] = (1-\lambda)(\lambda^2 - 2\lambda + 5) = 0.$$

Hence

$$\lambda = 1 \quad \text{or} \quad 1 \pm 2i. \quad \textbf{[1pt]}$$

For $\lambda = 1$, the corresponding eigenvector $\vec{\xi}$ solves

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \vec{0}.$$

Take $\xi_1 = 2$, the above identity gives a eigenvector $\vec{\xi} = (2,-3,2)^T$ and a solution to the linear system

$$\vec{x}^{(1)}(t) = e^{t} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}. \quad \textbf{[1pt]}$$

For $\lambda = 1 + 2i$, the corresponding eigenvector $\vec{\eta}$ solves

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \vec{0}.$$

Take $\eta_2 = 1$, one has $\eta_1 = 0, \eta_3 = -i$, hence

$$\vec{x} = e^t(\cos 2t + i \sin 2t) \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} = e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + ie^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}$$

is a solution. And since the coefficient matrix $\mathbb{A}$ is real, so the real and imaginary parts of the above solution are also solutions, i.e.

$$\vec{x}^{(2)}(t) = e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix}; \quad \vec{x}^{(3)}(t) = e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}. \quad \textbf{[1pt]}$$

In conclusion, the general solution is

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}$$

for some constants c_1, c_2, c_3.

(c) Any eigenvalue λ of $\mathbb{A}$ solves

$$0 = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = -(\lambda+1)(\lambda-2)^2 \iff \lambda = -1, 2, \quad \textbf{[1pt]}$$

where $\lambda = 2$ is a repeated eigenvalue. For $\lambda = -1$, the corresponding eigenvector $\vec{\xi}$ satisfies

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \vec{0}.$$

Take $\xi_3 = 2$, then $\xi_2 = 4$ and $\xi_1 = -3$. Therefore a solution to the linear system is given by

$$\vec{x}^{(1)}(t) = e^{-t} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}. \quad \textbf{[0.5pt]}$$

Secondly, for $\lambda = 2$, the corresponding eigenvector $\vec{\eta}$ is given by

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \vec{0}.$$

Taking $\eta_2 = 1$ implies $\eta_1 = 0$ and $\eta_3 = -1$. This gives another independent solution

$$\vec{x}^{(2)}(t) = \vec{\eta} e^{2t} = e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \quad \textbf{[0.5pt]}$$

In order to find the third one, we first find the other generalized eigenvector $\vec{\theta}$ corresponding to the eigenvalue $\lambda = 2$ of $\mathbb{A}$.

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = (\mathbb{A} - 2\mathbb{I})\vec{\theta} = \vec{\eta} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Taking $\theta_2 = 1$ gives $\theta_1 = 1$ and $\theta_3 = 0$. Hence $\vec{\theta} = (1, 1, 0)^T$ and the third independent solution could be

$$\vec{x}^{(3)}(t) = e^{2t}\big(t\vec{\eta} + \vec{\theta}\big) = e^{2t} \begin{pmatrix} 1 \\ t+1 \\ -t \end{pmatrix}. \quad \textbf{[1pt]}$$

In conclusion, the general solution is

$$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ t+1 \\ -t \end{pmatrix}$$

for some constants c_1, c_2, c_3.

4. (a) (i) Denote $\vec{H}(t) = \vec{H}^{(1)}(t) + \vec{H}^{(2)}(t)$ with $\vec{H}^{(1)}(t) = (e^{2t}, 0)^t$ and $\vec{H}^{(2)}(t) = (0, t^2)^t$. Let the particular solution $\vec{Y}(t) = \vec{Y}^{(1)}(t) + \vec{Y}^{(2)}(t)$, where $\vec{Y}^{(1)}(t)$ and $\vec{Y}^{(2)}(t)$ solve

$$\begin{cases} \dfrac{d}{dt}\vec{Y}^{(1)}(t) = \mathbb{A}\vec{Y}^{(1)}(t) + \vec{H}^{(1)}(t); \\ \dfrac{d}{dt}\vec{Y}^{(2)}(t) = \mathbb{A}\vec{Y}^{(2)}(t) + \vec{H}^{(2)}(t). \end{cases}$$

Since the eigenvalues of $\mathbb{A}$ are given by

$$0 = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda - 2)^2 \quad \Longleftrightarrow \quad \lambda = 2.$$

Take $\vec{Y}^{(1)} = (\vec{a}t^2 + \vec{b}t + \vec{c})e^{2t}$ **[0.5pt]**, the equation satisfied by $\vec{Y}^{(1)}(t)$ implies

$$2\vec{a}t^2 + 2\vec{a}t + 2\vec{b}t + \vec{b} + 2\vec{c} = \mathbb{A}(\vec{a}t^2 + \vec{b}t + \vec{c}) + (1,0)^T.$$

This can be rewritten as

$$(\mathbb{A} - 2\mathbb{I})\vec{a}t^2 + \big[(\mathbb{A} - 2\mathbb{I})\vec{b} - 2\vec{a}\big]t + (\mathbb{A} - 2\mathbb{I})\vec{c} - \vec{b} + (1,0)^T = 0.$$

Hence we choose $\vec{a}, \vec{b}, \vec{c}$ satisfying

$$\begin{cases} (\mathbb{A} - 2\mathbb{I})\vec{a} = 0; \\ (\mathbb{A} - 2\mathbb{I})\vec{b} - 2\vec{a} = 0; \\ (\mathbb{A} - 2\mathbb{I})\vec{c} - \vec{b} + (1,0)^T = 0, \end{cases} \qquad \text{where} \quad \mathbb{A} - 2\mathbb{I} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

These equations give, respectively,

$$\begin{cases} a_1 + a_2 = 0; \\ b_1 + b_2 = -2a_1 = 2a_2; \\ -c_1 - c_2 = b_1 - 1 \quad \text{and} \quad c_1 + c_2 = b_2. \end{cases}$$

The last two equations indicate $b_1 - 1 = -b_2$, which combined with the second equation implies $a_1 = -1/2$. And then $a_2 = 1/2$ by the first equation. Take $b_1 = 0$, then $b_2 = 1$; Take $c_1 = 0$, then $c_2 = 1$. Therefore

$$\vec{Y}^{(1)}(t) = e^{2t}\left[\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]. \quad \textbf{[1pt]} \tag{3}$$

Now we proceed to find $\vec{Y}^{(2)}(t)$. Take $\vec{Y}^{(2)}(t) = \vec{u}t^2 + \vec{v}t + \vec{w}$ **[0.5pt]**, then

$$2\vec{u}t + \vec{v} = \mathbb{A}(\vec{u}t^2 + \vec{v}t + \vec{w}) + t^2(0,1)^T.$$

This identity can be rewritten as

$$\left[\mathbb{A}\vec{u} + (0,1)^T\right]t^2 + \left(\mathbb{A}\vec{v} - 2\vec{u}\right)t + \mathbb{A}\vec{w} - \vec{v} = 0.$$

The above identity holds after imposing

$$\begin{cases} \mathbb{A}\vec{u} + (0,1)^T = 0; \\ \mathbb{A}\vec{v} - 2\vec{u} = 0; \\ \mathbb{A}\vec{w} - \vec{v} = 0. \end{cases}$$

Since $\mathbb{A}$ is non degenerate, one can solve the above equations one by one and get

$$\vec{u} = -\frac{1}{4}(1,1)^T, \quad \vec{v} = \left(-\frac{1}{2}, 0\right)^T, \quad \vec{w} = \frac{1}{8}(-3,1)^T.$$

Therefore

$$\vec{Y}^{(2)}(t) = -\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \frac{1}{8}\begin{pmatrix} -3 \\ 1 \end{pmatrix}. \quad \textbf{[1pt]} \tag{4}$$

As a consequence, with $\vec{Y}^{(1)}(t)$ and $\vec{Y}^{(2)}(t)$ defined in (3) and (4) respectively, a particular solution is given by

$$\vec{Y}(t) = \vec{Y}^{(1)}(t) + \vec{Y}^{(2)}(t).$$

(ii) Let $\vec{Y}(t) = \vec{Y}^{(1)}(t) + \vec{Y}^{(2)}(t)$ be a particular solution with

$$\begin{cases} \dfrac{d}{dt}\vec{Y}^{(1)}(t) = \mathbb{A}\vec{Y}^{(1)}(t) + e^t(1,0)^T; \\ \dfrac{d}{dt}\vec{Y}^{(2)}(t) = \mathbb{A}\vec{Y}^{(2)}(t) + \sqrt{3}e^{-t}(0,1)^T \end{cases}$$

Observe that both 1 and -1 are not eigenvalues of $\mathbb{A}$. Take $\vec{Y}^{(1)}(t) = \vec{a}e^t$, $\vec{Y}^{(2)}(t) = \vec{b}e^{-t}$ **[1pt]**, then

$$\begin{cases} \vec{a} = \mathbb{A}\vec{a} + (1,0)^T; \\ -\vec{b} = \mathbb{A}\vec{b} + \sqrt{3}(0,1)^T. \end{cases}$$

Solving the equalities yields

$$\vec{a} = \begin{pmatrix} -\dfrac{\sqrt{2}}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\dfrac{\sqrt{6}}{3} \\ -\dfrac{\sqrt{2}}{2} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}.$$

In conclusion, a particular solution is given by

$$\vec{Y}(t) = \vec{Y}^{(1)}(t) + \vec{Y}^{(2)}(t) = e^t \begin{pmatrix} -\dfrac{\sqrt{6}}{3} \\ -\dfrac{\sqrt{2}}{2} \end{pmatrix} + e^{-t} \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}. \quad \textbf{[2pts]}$$

(b) (i) Firstly, the eigenvalues of $\mathbb{A}$ are given by

$$0 = \begin{vmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-1) - 4 = (\lambda+3)(\lambda-2) \Longleftrightarrow \lambda = -3, 2. \quad \textbf{[1pt]}$$

For $\lambda = -3$, the associated eigenvector $\vec{\xi}$ solves

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \vec{\xi} = \vec{0}.$$

One may take $\vec{\xi} = (1, -1)^T$ and this gives one solution

$$\vec{x}^{(1)}(t) = e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \textbf{[0.5pt]}$$

For $\lambda = 2$, arguing similarly, the associated eigenvector can be taken to be $\vec{\eta} = (4, 1)^T$ and this gives another independent solution

$$\vec{x}^{(2)}(t) = e^{2t} \begin{pmatrix} 4 \\ 1 \end{pmatrix}. \quad \textbf{[0.5pt]}$$

Consequently, the fundamental set of solutions is

$$\left\{ e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e^{2t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}. \quad \textbf{[0.5pt]}$$

And the fundamental matrix is

$$\mathbb{F}(t) = \begin{pmatrix} e^{-3t} & 4e^{2t} \\ -e^{-3t} & e^{2t} \end{pmatrix}. \quad \textbf{[0.5pt]}$$

(ii) Let $\mathbb{C}$ be a constant matrix such that

$$\mathbb{G}(t) = \mathbb{F}(t)\mathbb{C}.$$

Since $\mathbb{G}(0) = \mathbb{I}$, the above identity implies $\mathbb{C} = \mathbb{F}^{-1}(0)$. Direct calculations give

$$\mathbb{F}(0) = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix} \quad \text{and then} \quad \mathbb{F}^{-1}(0) = \frac{1}{5} \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}.$$

Therefore **[2pts]**

$$\mathbb{G}(t) = \mathbb{F}(t)\mathbb{C} = \frac{1}{5} \begin{pmatrix} e^{-3t} & 4e^{2t} \\ -e^{-3t} & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} e^{-3t} + 4e^{2t} & -4e^{-3t} + 4e^{2t} \\ -e^{-3t} + e^{2t} & 4e^{-3t} + e^{2t} \end{pmatrix}.$$

(iii) Let $\vec{Y}(t) = \mathbb{F}(t)\vec{u}(t)$. Note that $\det(\mathbb{F}) = 5e^{-t}$, then

$$\vec{u}\,'(t) = \mathbb{F}^{-1}(t)\vec{H}(t) = \frac{e^t}{5}\begin{pmatrix} e^{2t} & -4e^{2t} \\ e^{-3t} & e^{-3t} \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} e^t + 8e^{4t} \\ e^{-4t} - 2e^{-t} \end{pmatrix}$$

Integrating directly gives

$$\vec{u}(t) = \frac{1}{5}\begin{pmatrix} e^t + 2e^{4t} \\ -e^{-4t}/4 + 2e^{-t}. \end{pmatrix}$$

Consequently, one has the following particular solution

$$\vec{Y}(t) = \mathbb{F}(t)\vec{u}(t) = \begin{pmatrix} 2e^t \\ -e^{-2t}/4 \end{pmatrix}. \quad \textbf{[5pts]}$$

(iv) Firstly, the eigenvalues of matrix $\mathbb{A}$ are given by

$$\begin{vmatrix} -5/4 - \lambda & 3/4 \\ 3/4 & -5/4 - \lambda \end{vmatrix} = (\lambda + 1/2)(\lambda + 2) = 0.$$

For $\lambda = -1/2$, the corresponding eigenvector solves

$$\begin{pmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{pmatrix}\vec{\xi} = \vec{0}.$$

One may take $\vec{\xi} = (1,1)^T$. For $\lambda = -2$, the corresponding eigenvector can be taken to be $\vec{\eta} = (1,-1)^T$. Hence the fundamental set of solutions to the homogeneous system is

$$\left\{ e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

And the fundamental matrix is

$$\mathbb{F}(t) = \begin{pmatrix} e^{-t/2} & e^{-2t} \\ e^{-t/2} & -e^{-2t} \end{pmatrix}. \quad \textbf{[3pts]}$$

$$\mathbb{F}(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and then} \quad \mathbb{F}^{-1}(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore **[2pts]**

$$\mathbb{G}(t) = \mathbb{F}(t)\mathbb{F}^{-1}(0) = \frac{1}{2}\begin{pmatrix} e^{-t/2} & e^{-2t} \\ e^{-t/2} & -e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-t/2} + e^{-2t} & e^{-t/2} - e^{-2t} \\ e^{-t/2} - e^{-2t} & e^{-t/2} + e^{-2t} \end{pmatrix}.$$

Lastly, we derive a particular solution $\vec{Y}(t)$ to the inhomogeneous problem. Let $\vec{Y}(t) = \mathbb{F}(t)\vec{u}(t)$, then

$$\vec{u}\,'(t) = \mathbb{F}^{-1}(t)\vec{H}(t) = -\frac{e^{5t/2}}{2}\begin{pmatrix} -e^{-2t} & -e^{-2t} \\ -e^{-t/2} & e^{-t/2} \end{pmatrix}\begin{pmatrix} 4t \\ e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4te^{t/2} + e^{3t/2} \\ 4te^{2t} - e^{3t} \end{pmatrix}.$$

Integrating directly and choosing the integrating constant to be zero lead to

$$\vec{u}(t) = \begin{pmatrix} (4t-8)e^{t/2} + e^{3t/2}/3 \\ (t - 1/2)e^{2t} - e^{3t}/6 \end{pmatrix},$$

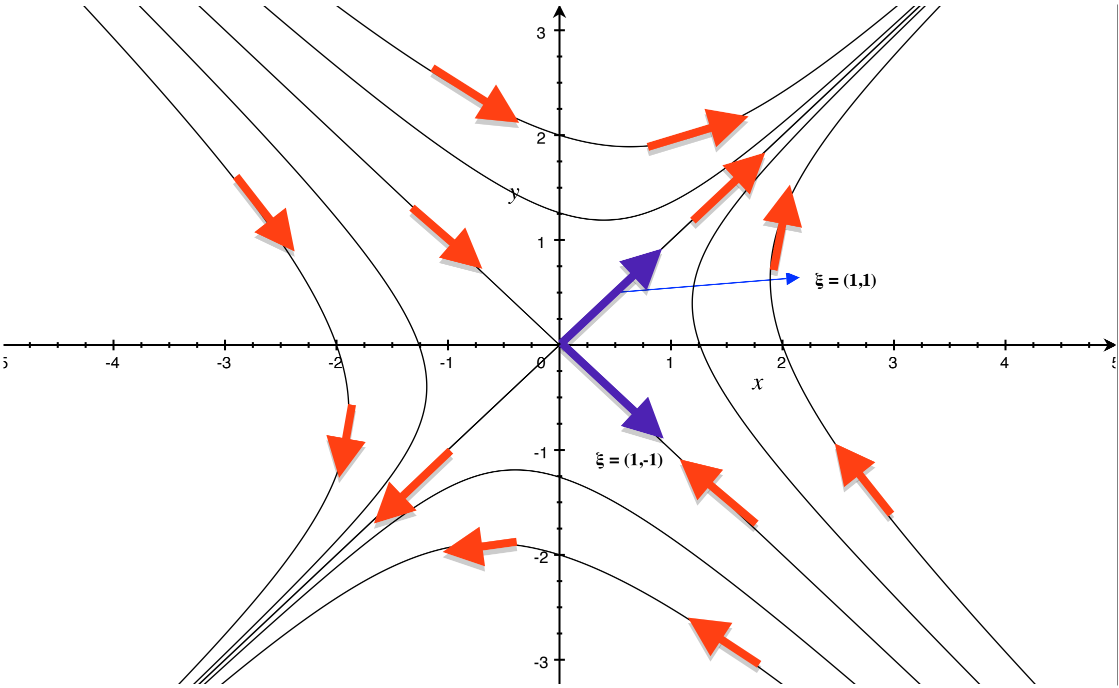
and then

$$\vec{Y}(t) = \mathbb{F}(t)\vec{u}(t) = \begin{pmatrix} 5t - 17/2 + e^t/6 \\ 3t - 15/2 + e^t/2 \end{pmatrix}. \quad \textbf{[5pts]}$$

5. Let $\vec{Y}(t) = (x, y)^T$.

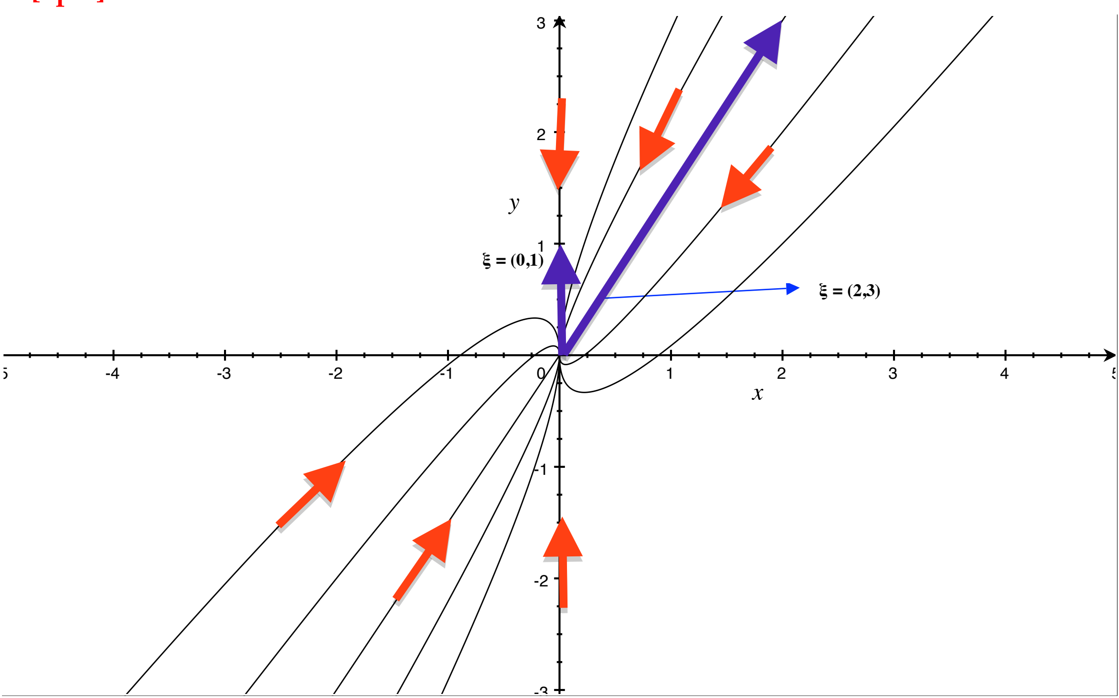
(i) $\vec{0}$ is a saddle point. **[1pt]**

[2pts]



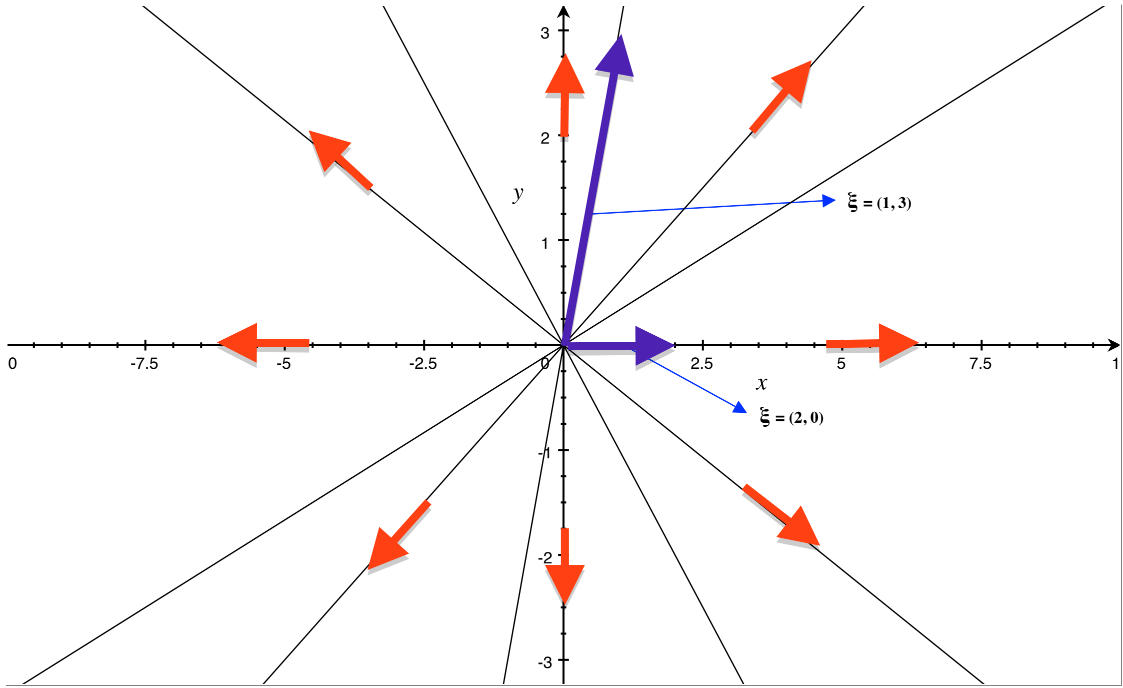
(ii) $\vec{0}$ is a node.**[1pt]**

[2pts]



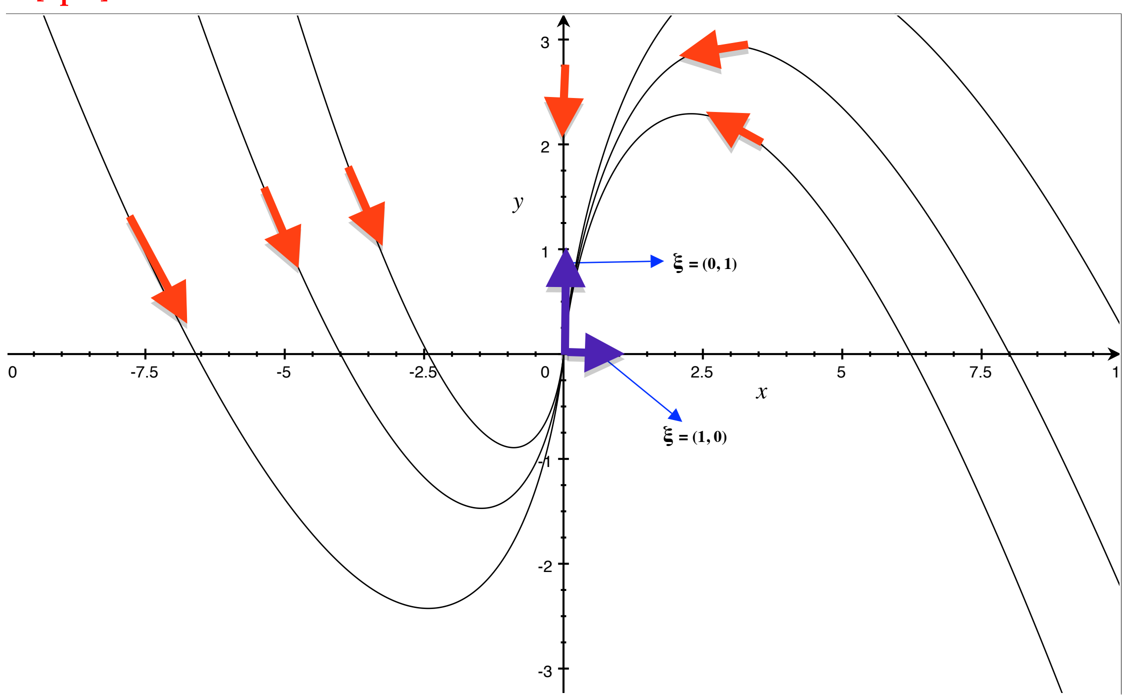
(iii) $\vec{0}$ is a (proper) node.**[1pt]**

[2pts]



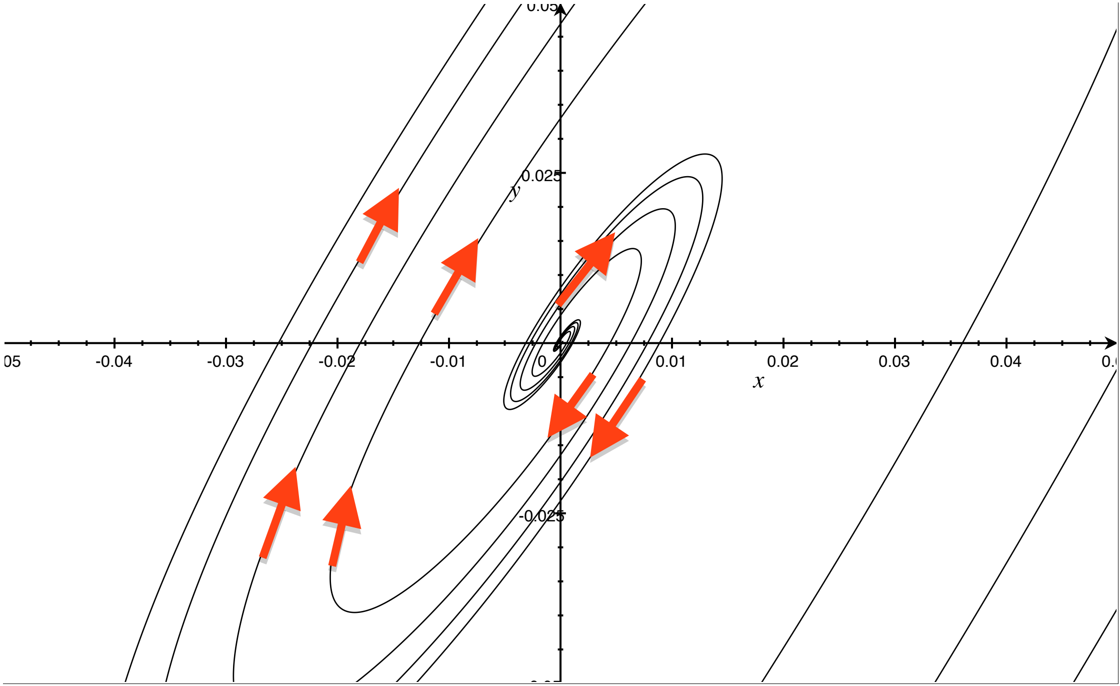
(iv) $\vec{0}$ is a(n) (improper) node.**[1pt]**

[2pts]



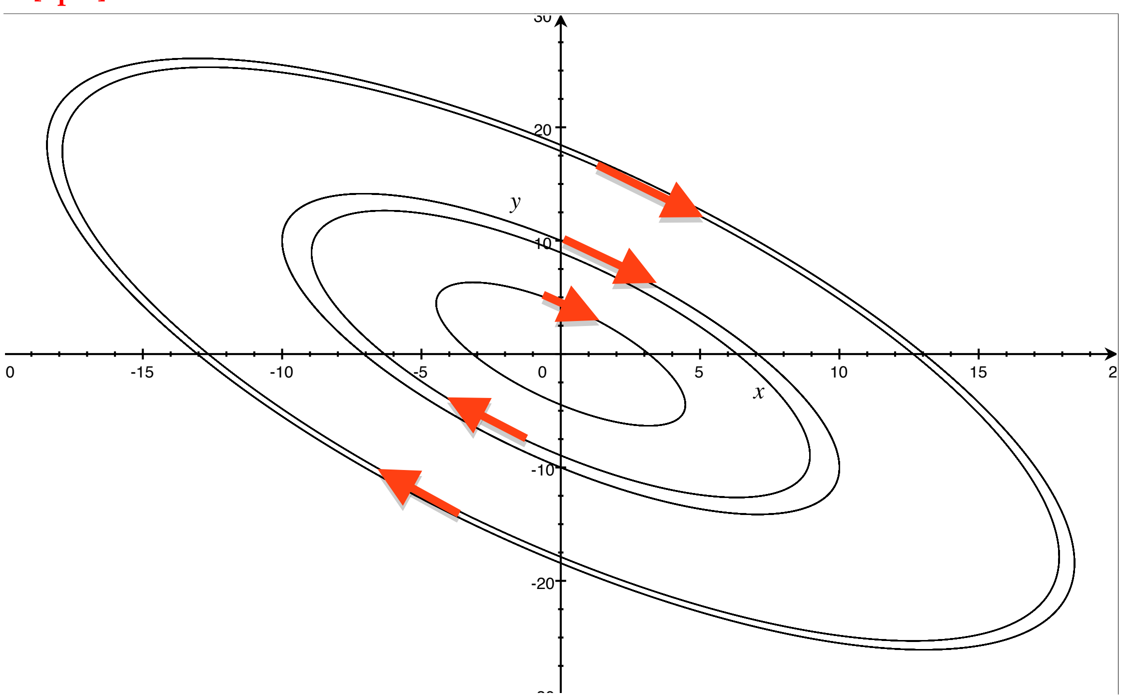
(v) $\vec{0}$ is a spiral point.**[1pt]**

[2pts]



(vi) $\vec{0}$ is a center point.**[1pt]**

[2pts]



6. (a) Direct computations show

$$\text{(i) } \mathbb{J} = \begin{pmatrix} 8-5y & -5t \\ -3y & 3-3t-2y \end{pmatrix}; \textbf{[2pts]} \quad \text{(ii) } \mathbb{J} = \begin{pmatrix} \sin y & (2+t)\cos y \\ 9(-\sin t + \cos t)e^t & 0 \end{pmatrix}; \textbf{[2pts]}$$

and

$$\text{(iii) } \mathbb{J} = \begin{pmatrix} 9 & 12y & 0 \\ y/t & \ln t + \sin y & 0 \\ \sinh(t+y) & \sinh(t+y) & 0 \end{pmatrix}. \textbf{[2pts]}$$

(b) It is clear that $(x+2y^2)|_{(0,0)} = 0$ and $(x+y)|_{(0,0)} = 0$, hence $\vec{0}$ is a critical point. The Jacobian matrix at $\vec{0}$ is

$$\mathbb{J} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad \textbf{[1pt]}$$

Then as $r := \sqrt{x^2 + y^2}$ goes to zero,

$$\frac{1}{r}\left\| \begin{pmatrix} x + 2y^2 \\ x + y \end{pmatrix} - \mathbb{J}\begin{pmatrix} x \\ y \end{pmatrix} \right\| = \frac{1}{r}\left\| \begin{pmatrix} 2y^2 \\ 0 \end{pmatrix} \right\| \leqslant 2r \to 0.$$

Consequently the system is locally linear near the critical point $\vec{0}$. **[2pts]**
Since $\mathbb{J}$ is a matrix with a real positive eigenvalue $\lambda = 1$ associated with only one independent eigenvector $\vec{\xi} = (0,1)^T$. Therefore this locally linear system is unstable **[1pt]** , and $\vec{0}$ is a(n) (improper) node **[1pt]**.

(c) (i) To find the critical points, we need solve

$$\begin{cases} x(1-2x-y) = 0; \\ y(-2+6x) = 0. \end{cases} \iff \begin{cases} x = 0 \quad \text{or} \quad 1 - 2x - y = 0; \\ y = 0 \quad \text{or} \quad -2 + 6x = 0. \end{cases}$$

Short derivations show that there are three critical points:

$$\vec{X}_{1,*} = (0,0)^T, \textbf{[1pt]} \quad \vec{X}_{2,*} = \left(\frac{1}{2}, 0\right) \textbf{[1pt]} \quad \text{and} \quad \vec{X}_{3,*} = \left(\frac{1}{3}, \frac{1}{3}\right). \textbf{[1pt]}$$

(ii) The Jacobian matrix is given by

$$\mathbb{J}(x,y) = \begin{pmatrix} 1 - 4x - y & -x \\ 6y & -2 + 6x \end{pmatrix}.$$

For the critical point $\vec{X}_{1,*} = \vec{0}$, since

$$\vec{g}^{(1)}(x,y) = \vec{F}(x,y) - \mathbb{J}(0,0)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(1-2x-y) \\ y(-2+6x) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} -2x^2 - xy \\ 6xy \end{pmatrix}$$

and by $|xy| \leqslant r_1^2/2 \leqslant r_1^2$, where $r_1^2 = x^2 + y^2$. Consequently,

$$\|\vec{g}^{(1)}\|/r_1 \leqslant \left[(2r_1^2 + r_1^2)^2 + (3r_1^2)^2\right]^{1/2} \Big/ r_1 = 3\sqrt{2}\, r_1 \longrightarrow 0 \quad \text{as } r_1 \text{ goes to } 0. \quad \textbf{[3pts]}$$

Hence the system (1) is locally linear at the critical point $\vec{X}_{1,*} = (0,0)^T$.
Secondly, for the critical point $\vec{X}_{2,*} = (\frac{1}{2}, 0)$, since

$$\vec{g}^{(2)}(x,y) = \begin{pmatrix} x(1-2x-y) \\ y(-2+6x) \end{pmatrix} - \begin{pmatrix} -1 & -1/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x - 1/2 \\ y \end{pmatrix}$$
$$= \begin{pmatrix} -2(x-1/2)^2 - y(x-1/2) \\ 6y(x-1/2) \end{pmatrix}.$$

Denote $r_2 = \sqrt{(x-1/2)^2 + y^2}$. Similarly,

$$\|\vec{g}^{(2)}\|/r_2 \leqslant \left[(2r_2^2 + r_2^2)^2 + (3r_2^2)^2\right]^{1/2} \Big/ r_2 = 3\sqrt{2}\, r_2 \longrightarrow 0 \quad \text{as } r_2 \text{ goes to } 0. \quad \textbf{[3pts]}$$

Hence the system (1) is locally linear at the critical point $\vec{X}_{2,*} = (1/2, 0)^T$.
Lastly, for the critical point $\vec{X}_{3,*} = (\frac{1}{3}, \frac{1}{3})$, since

$$\vec{g}^{(3)}(x,y) = \begin{pmatrix} x(1-2x-y) \\ y(-2+6x) \end{pmatrix} - \begin{pmatrix} -2/3 & -1/3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x-1/3 \\ y-1/3 \end{pmatrix}$$
$$= \begin{pmatrix} -2(x-1/3)^2 - (x-1/3)(y-1/3) \\ 6(x-1/3)(y-1/3) \end{pmatrix}.$$

Denote $r_3 = \sqrt{(x-1/3)^2 + (y-1/3)^2}$. Similarly,

$$\|\vec{g}^{(3)}\|/r_3 \leqslant \left[(2r_3^2 + r_3^2)^2 + (3r_3^2)^2\right]^{1/2} \Big/ r_3 = 3\sqrt{2}\, r_3 \longrightarrow 0 \quad \text{as } r_3 \text{ goes to } 0. \quad \textbf{[3pts]}$$

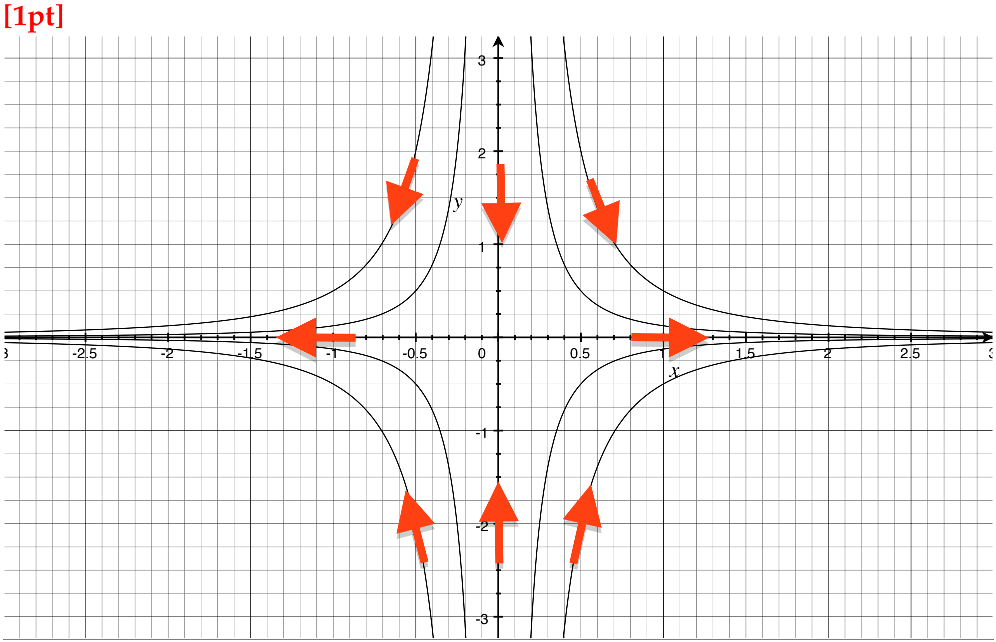
Hence the system (1) is locally linear at the critical point $\vec{X}_{3,*} = (1/3, 1/3)^T$.

(iii) For $\vec{X}_{1,*} = (0,0)^T$, the Jacobian matrix is

$$\mathbb{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},$$

whose eigenvalues are $\lambda = 1$ or -2. Hence the critical point $\vec{X}_{1,*} = (0,0)^T$ is a saddle point **[1pt]** and unstable **[1pt]**. Near this critical point, the phase portrait is like the following.

[1pt]

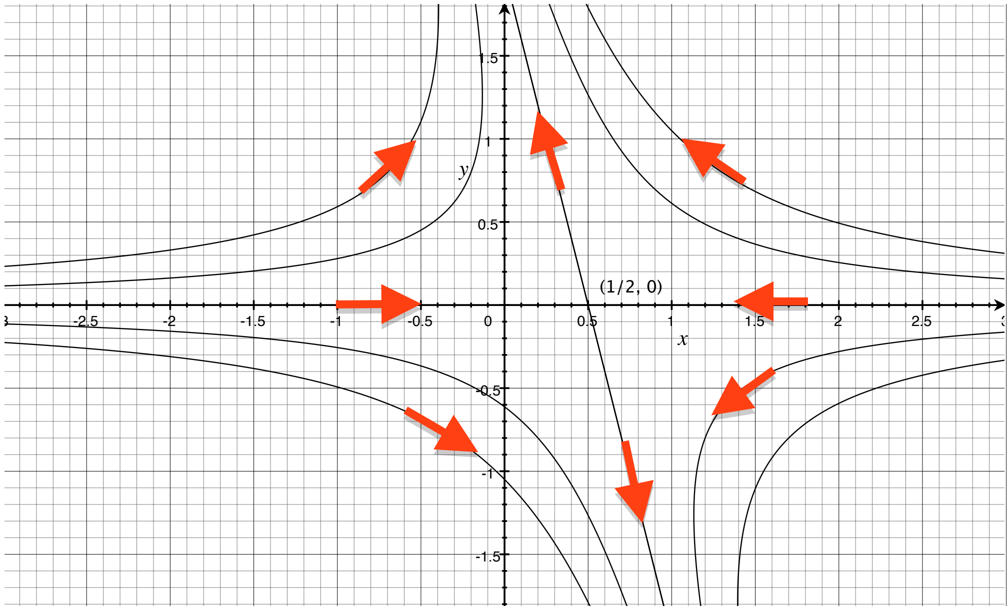


For $\vec{X}_{2,*} = (1/2, 0)^T$, the Jacobian matrix is

$$\mathbb{J}(1/2,0) = \begin{pmatrix} -1 & -1/2 \\ 0 & 1 \end{pmatrix},$$

whose eigenvalues are $\lambda = 1$ or -1. Hence the critical point $\vec{X}_{2,*} = (1/2, 0)^T$ is a saddle point **[1pt]** and unstable **[1pt]**. Near this critical point $\vec{X}_{2,*}$, the phase portrait is as follows.

[1pt]

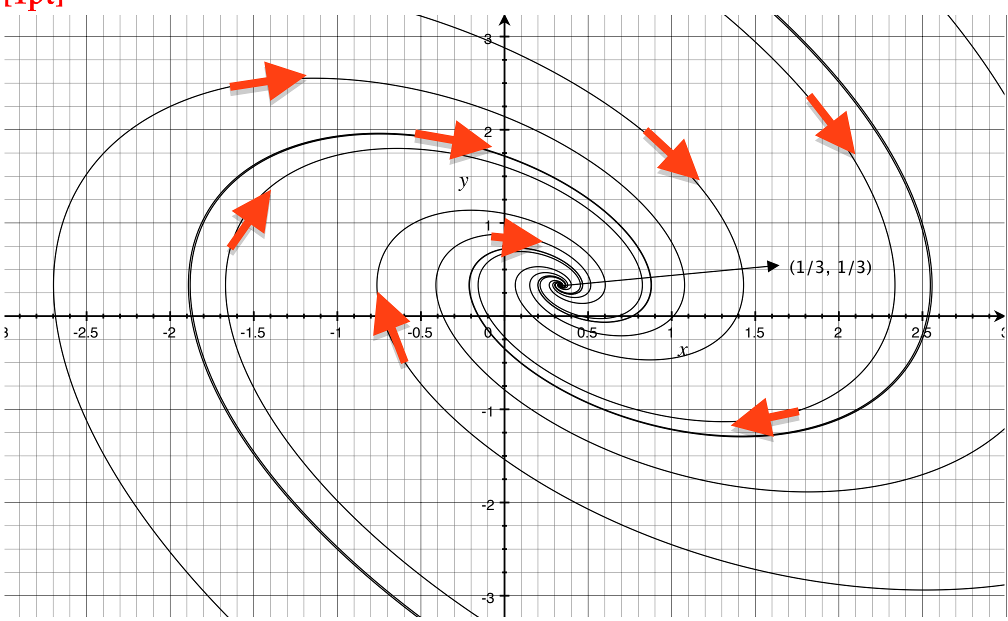


Lastly, for $\vec{X}_{3,*} = (1/3, 1/3)^T$, the Jacobian matrix is

$$\mathbb{J}(1/3, 1/3) = \begin{pmatrix} -2/3 & -1/3 \\ 2 & 0 \end{pmatrix},$$

whose eigenvalues are $\lambda = -1 \pm \sqrt{5}i$. Hence the critical point $\vec{X}_{3,*} = (1/3, 1/3)^T$ is a spiral point **[1pt]** and stable **[1pt]**. Near this critical point $\vec{X}_{3,*}$, the phase portrait is as follows.

[1pt]



7. (a) It is clear that $V_1(x, y) > 0$ for $(x, y) \neq (0, 0)$, and $V_1(0, 0) = 0$, hence $V_1(x, y)$ is positive definite. **[1pt]** Note that $|ab| \leqslant (a^2 + b^2)/2$, one easily check that for $(x, y) \neq (0, 0)$,

$$V_2(x, y) \geqslant 2x^2 + y^2 - \frac{x^2 + y^2}{2} = \frac{3x^2}{2} + \frac{y^2}{2} > 0;$$

$$V_3(x, y) \leqslant -x^2 - 7y^2 + \frac{x^2}{2} + 2y^2 = -\frac{x^2}{2} - 5y^2 < 0.$$

And observe that $V_2(0, 0) = 0$ and $V_3(0, 0) = 0$, therefore $V_2(x, y)$ is positive definite **[1pt]** and $V_3(x, y)$ is negative definite **[1pt]**.

(b) Obviously, $V(x,y)$ is positive definite since $V(0,0) = 0$ and $V(x,y) = a(x^2+y^2) > 0$ $(a>0)$ for $(x,y) \neq (0,0)$.**[2pts]** The remaining thing is to verify $\frac{d}{dt}V(x(t),y(t))$ is positive or negative definite. First, for (i) and $(x,y) \neq (0,0)$,

$$\frac{d}{dt}V(x(t),y(t)) = 2a(xx' + yy') = 2a(-x^4 - y^4) < 0.$$

This, combined with the fact $\frac{d}{dt}V|_{(\vec{0})} = 0$, implies the derivative of V w.r.t. t is negative definite.**[2pts]** Second, for (ii) and $(x,y) \neq (0,0)$,

$$\frac{d}{dt}V(x(t),y(t)) = 2a(xx' + yy') = 2a(x^6 + y^4) > 0,$$

which implies similarly that the derivative of V w.r.t. t is positive definite.**[2pts]** In conclusion, the function $V(x,y) = a(x^2+y^2)$ with $a > 0$ is a Liapunov function for both systems (i) and (ii).

(c) Note that $V(x,y)$ is positive definite. For (b)(i), $\frac{d}{dt}V(x(t),y(t))$ is negative definite, hence by Liapunov's theorem, $\vec{0}$ is a stable critical point;**[2pts]** For (b)(ii), $\frac{d}{dt}V(x(t),y(t))$ is positive definite, hence by Liapunov's theorem, $\vec{0}$ is an unstable critical point.**[2pts]**

(d) When $V(x,y) = ax^2 + cy^2$ for some constants a and c, computing

$$\begin{aligned}\frac{d}{dt}V(x(t),y(t)) &= 2axx' + 2cyy' = 2a(-xy - x^4 - x^2y^2) + 2c(xy - y^2x^2 - y^4)\\ &= -2ax^4 - 2cy^4 - 2(a-c)xy - 2(a+c)x^2y^2.\end{aligned}$$

To construct a Liapunov function, we take $a = c = 1$ and then $V(x,y) = x^2 + y^2$, which is obviously positive definite.**[3pts]** And since for $(x,y) \neq (0,0)$,

$$\frac{d}{dt}V(x(t),y(t)) = -2x^4 - 2y^4 - 4x^2y^2 = -2(x^2+y^2)^2 = -2V^2(x(t),y(t)) < 0.$$

This implies easily $\frac{d}{dt}V(x(t),y(t))$ is negative definite.**[3pts]** Therefore $V(x,y) = x^2+y^2$ is a Liapunov function for the given system. By Liapunov's theorem, $\vec{0}$ is an asymptotic stable critical point. **[2pts]**