MATH3230A Numerical Analysis

Tutorial 9 with solution

1 Recall:

Numerical Integration: We are going to estimate the integral

$$\int_a^b f(x)dx$$

1. **Newton-Cotes Quadrature Rule**

   (a) Idea: For equally spaced set of points x_i, to find $\alpha_0, \alpha_1, \cdots, \alpha_n$ such that for any polynomial of degree $\leq n$, we have

   $$\int_a^b p(x)dx = \alpha_0 p(x_0) + \alpha_1 p(x_1) + \cdots + \alpha_n p(x_n)$$

   By uniquness of polynomial interpolation, we may pick $\alpha_i = \int_a^b l_i(x)dx$ using Lagrange polynomials.

   (b) For $n = 2$, we have the Simpson's Rule:

   $$\int_a^b f(x)dx \approx \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}$$

   The corresponding error:

   $$|\int_a^b f(x)dx - \frac{b-a}{6}\{f(a) + 4f(\frac{a+b}{2}) + f(b)\}| \leq \frac{49}{2880}K(b-a)^5$$

   where $max_{x\in[a,b]}|f^{(4)}(x)| \leq K$

   (c) The Composite Simpson's Rule is

   $$\int_a^b f(x)dx \approx \frac{h}{6}\sum_{i=1}^{n}\left\{f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right\}$$

   The corresponding error:

   $$|\int_a^b f(x) - \frac{h}{6}\sum_{i=1}^{n}\left\{f(x_{i-1} + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right\}| = \frac{49K}{2880}(b-a)h^4$$

2. **Gaussian Quadrature Rule**

   (a) The Gaussian Quadrature rule satisfies

   $$\int_a^b p(x)dx = \sum_{i=0}^{n}\alpha_i p(x_i) \text{ for all polynomials of degree } \leq 2n+1$$

   Here we need to determine both points x_i and the coefficients α_i. In total there are $(2n+2)$ unknowns. 2 point Gaussian quadrature rule on $[-1, 1]$:

   $$\int_{-1}^1 f(x)dx \approx f(\frac{-1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$$

(b) Legendre polynomial $P_n(x)$ is defined recursively by:

$$P_0(x) = 1, \quad P_1(x) = x, \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

The procedure of deriving Gauss-Legendre quadrature rule is as:

i. Let $\{x_i\}$ be roots of $P_{n+1}(x)$.

ii. Set $\alpha_i = \int_{-1}^{1} l_i(x)dx = \int_{-1}^{1} \prod_{j\neq i, j=0}^{n} \frac{x-x_j}{x_i-x_j} dx$.

iii. Then the rule is $\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} \alpha_i f(x_i)$

The Gauss-Legendre quadrature rule is exact for polynomial with degree $\leq 2n+1$.

(c) With Gauss-Legendre quadrature defined above, suppose $f \in C^{2n+2}[-1,1]$, $max_{x\in[-1,1]}|f^{2n+2}(x)| \leq K$ we have:

$$|\int_{-1}^{1} f(x)dx - \sum_{i=0}^{n} \alpha_i f(x_i)| \leq \frac{K}{(2n+2)!} \int_{-1}^{1} (x-x_0)^2 \cdots (x-x_n)^2 dx$$

(d) For $f : [a,b] \to \mathbb{R}$, we apply the linear transformation: $y = h(x) = \frac{a+b}{2} + \frac{b-a}{2}x$.

2 Exercises:

Please submit solutions of problems with star(*) before 6:30PM on Wednesday and finish the rest by yourself.

1. * Consider the following questions:

(a) Find $\alpha_0, \alpha_1, \alpha_2$ and α_3 such that the quadrature formula

$$\int_{-1}^{1} g(t)dt \approx \alpha_0 g(-1) + \alpha_1 g(-\frac{1}{3}) + \alpha_2 g(\frac{1}{3}) + \alpha_3 g(1)$$

is exact for all polynomials of degree less than or equal to 3.

(b) Using the quadrature formula obtained in (a), derive the corresponding quadrature formula for computing

$$\int_a^b g(x)dx$$

(Using the transformation introduced in P127)

Solution. (a) Let $g(t) = 1, t, t^2, t^3$ respectively, then we have

$$\begin{aligned}
\int_{-1}^{1} 1dt = 2 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
\int_{-1}^{1} tdt = 0 &= \alpha_0(-1) + \alpha_1(-\frac{1}{3}) + \alpha_2(\frac{1}{3}) + \alpha_3 \\
\int_{-1}^{1} t^2dt = \frac{2}{3} &= \alpha_0(-1)^2 + \alpha_1(-\frac{1}{3})^2 + \alpha_2(\frac{1}{3})^2 + \alpha_3 \\
\int_{-1}^{1} t^3dt = 0 &= \alpha_0(-1)^3 + \alpha_1(-\frac{1}{3})^3 + \alpha_2(\frac{1}{3})^3 + \alpha_3
\end{aligned}$$

Solving the above equations, we have

$$\alpha_0 = \frac{1}{4}, \alpha_1 = \frac{3}{4}, \alpha_2 = \frac{3}{4} \quad and \quad \alpha_3 = \frac{1}{4}.$$

(b) Let $x = a + \frac{b-a}{2}(t+1)$. We have

$$\begin{aligned}
\int_a^b g(x)dx &= \int_{-1}^{1} g[a + \frac{b-a}{2}(t+1)]\frac{(b-a)}{2}dt \\
&\approx \frac{(b-a)}{2}[\frac{1}{4}g(a) + \frac{3}{4}(a + \frac{(b-a)}{3}) + \frac{3}{4}g(a + \frac{2(b-a)}{3}) + \frac{1}{4}g(b)] \\
&= \frac{b-a}{2}\left[\frac{1}{4}g(a) + \frac{3}{4}g\left(\frac{2a+b}{3}\right) + \frac{3}{4}g\left(\frac{a+2b}{3}\right) + \frac{1}{4}g(b)\right]
\end{aligned}$$

$\square$

2. * Let $f(x)$ be a real function defined on $[0,1]$, $x_0 = 0$, $x_1 = \frac{1}{3}$ and $x_2 = 1$.

   (a) Consider the quadratic polynomial $p(x)$:

$$p(x) = \alpha_0(x - x_1)(x - x_2) + \alpha_1(x - x_0)(x - x_2) + \alpha_2(x - x_0)(x - x_1) \tag{1}$$

   Compute the integral

$$\int_0^1 p(x)dx,$$

   and write down the result explicitly in terms of α_0, α_1 and α_2.

   (b) If the polynomial (1) is the Lagrange interpolation of function $f(x)$, write down your result in (a) into the following form explicitly:

$$\int_0^1 p(x)dx = \alpha_0 f(x_0) + \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

   (c) Let f be a real function on $[0,1]$ and $\{\alpha_i\}_{i=0}^2$ be the coefficient above. If we approximate the integral

$$\int_0^1 f(x)dx$$

   by the formula

$$\int_0^1 f(x)dx \approx \alpha_0 f(x_0) + \alpha_1 f(x_1) + \alpha_2 f(x_2),$$

   show that this formula is exact for all polynomials of degree ≤ 2.

Solution. (a) A direct computation yields

$$p(x) = (\alpha_0 + \alpha_1 + \alpha_2)x^2 - (\frac{4\alpha_0}{3} + \alpha_1 + \frac{\alpha_2}{3})x + \frac{\alpha_0}{3}$$

Since $\int_0^1 x^2 dx = 1/3$ and $\int_0^1 x dx = 1/2$,

$$\int_0^1 p(x)dx = -\frac{\alpha_1}{6} + \frac{\alpha_2}{6}.$$

(b) If $p(x)$ is an interpolation of $f(x)$, we have

$$\alpha_1 = -\frac{9}{2} f(x_1) \text{ and } \alpha_2 = \frac{3}{2} f(x_2)$$

Then,

$$\int_0^1 p(x)dx = \frac{3f(x_1)}{4} + \frac{f(x_2)}{4}.$$

(c) It is easy to see that this formula is exact for $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$. Given a polynomial $p(x)$of degree ≤ 2, there exists α_0, α_1 and α_2 such that

$$p(x) = \alpha_0 f_0(x) + \alpha_1 f_1(x) + \alpha_2 f_2(x)$$

Therefore,

$$\int_0^1 p(x)dx = \sum_{i=0}^{2} \alpha_i \int_0^1 f_i(x)dx = \sum_{i=0}^{2} \alpha_i \left(\frac{3f_i(x_0)}{4} + \frac{f_i(x_2)}{4} \right) = \frac{3p(x_1)}{4} + \frac{p(x_2)}{4}.$$

$\square$

3. In the following exercise, we consider the Gauss-Legendre quadrature rule:

   (a) * Derive the Gauss-Legendre quadrature rule step by step with 3 nodal points in $[-1,1]$. (Recall $P_3(x) = \frac{1}{2}(5x^3 - 3x)$)

   (b) * Suppose $max_{x\in[-1,1]}|f^{(6)}(x)| \leq 9$, then compute the error estimates when using above Gauss-Legendre quadrature rule to approximate $\int_{-1}^{1} f(x)dx$.

   (c) Using the composite trapezoidal rule (with 3 nodal points), Simpson's rule, 3 points Gauss-Legendre quadrature rule separately to compute $\int_{-1}^{1} e^x dx$ and compare their accuracy.

Solution. (a) Solving $P_3(x)$, we find $x_0 = -\sqrt{\frac{3}{5}}, x_1 = 0, x_2 = \sqrt{\frac{3}{5}}$;

Then using $\alpha_i = \int_{-1}^{1} l_i(x)dx = \int_{-1}^{1} \prod_{j\neq i,j=0}^{2} \frac{x-x_j}{x_i-x_j}dx$, to get $\alpha_0 = \frac{5}{9}$, $\alpha_1 = \frac{8}{9}$, $\alpha_2 = \frac{5}{9}$.

Now the rule is $\int_{-1}^{1} f(x)dx \approx \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}})$.

(b)

$$\begin{aligned}
|\int_{-1}^{1} f(x)dx - (\frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}}))| &\leq \frac{9}{6!}\int_{-1}^{1}(x+\sqrt{\frac{3}{5}})^2 x^2 (x-\sqrt{\frac{3}{5}})^2 dx \\
&= \frac{1}{80}\int_{-1}^{1}(x^2-\frac{3}{5})^2 x^2 dx \\
&= \frac{1}{80}\int_{-1}^{1}(x^6-\frac{6}{5}x^4+\frac{9}{25}x^2)dx \\
&= \frac{1}{80}[\frac{x^7}{7}-\frac{6}{25}x^5+\frac{3}{25}x^3]_{-1}^{1} \\
&= \frac{1}{2250}
\end{aligned}$$

(c) i. Using composite trapezoidal rule:

$$\int_{-1}^{1} e^x dx = \frac{3}{2}(\frac{e^{-1}}{2} + e^0 + \frac{e^1}{2}) \approx 3.8146$$

ii. Using Simpson's rule:

$$\int_{-1}^{1} e^x dx = \frac{2}{6}(e^{-1} + 4e^0 + e^1) \approx 2.3621$$

iii. Using 3 points Gauss-Legendre rule:

$$\int_{-1}^{1} e^x dx = \frac{5}{9}exp(-\sqrt{\frac{3}{5}}) + \frac{8}{9}exp(0) + \frac{5}{9}exp(\sqrt{\frac{3}{5}}) \approx 2.3503$$

While the exact value with 5 digits of accuracy is 2.3504.

□