MATH3230A Numerical Analysis

Tutorial 7 with solution

1 Recall:

1. **Vandermonde interpolation:**
   Suppose we are given $n+1$ observation data:
   $$f_0 = f(x_0), f_1 = f(x_1), \ldots, f_{n+1} = f(x_n)$$
   where $x_i \neq x_j$ for all $i \neq j$. We determine a polynomial $p(x)$ of degree $\leq n$ such that
   $$p(x_i) = f_i, \quad i = 0, 1, \ldots, n$$
   Suppose $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots \alpha_n x^n$, we have
   $$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{1}$$
   where the coefficient matrix is called a Vandermonde matrix. Uniqueness of the polynomial $p(x)$ is guaranteed. But solving for the coefficients α_i is computationally expensive and it may be very ill-conditioned (large condition number).

2. **Lagrange interpolation:**
   Consider the following basis functions:
   $$l_j(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)} \tag{2}$$
   for $j = 0, 1, \cdots, n$. Note that $l_j(x_j) = 1$ and $l_j(x_i) = 0$ for all $i \neq j$. Then the following polynomial of degree $\leq n$
   $$L(x) = f_0 l_0(x) + f_1 l_1(x) + \cdots f_n l_n(x)$$
   will satisfy $L(x_i) = f_i$ for all $i = 0, 1, \cdots, n$.

3. **Newton form of interpolation:**
   Suppose $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Then we define the Divided difference as follows:
   The zeroth-order divided difference of $f(x)$ is
   $$f[x_0] = f(x_0), \quad f[x_1] = f(x_1), \cdots, f[x_n] = f(x_n)$$
   The first order divided difference of $f(x)$ is
   $$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \quad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}, \quad , \cdots ,$$

and similar we have the k-th order divided difference of $f(x)$

$$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots x_{k-1}]}{x_k - x_0}, \quad ,$$

The Newton form of interpolation of $f(x)$ is

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

4. **Error estimates of polynomial interpolations:**
Suppose $f \in C^{n+1}[a, b]$ and $p(x)$ is the polynomial interpolation of $f(x)$ at the $n+1$ distinct points:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

then for any $x \in [a, b]$, there exists a point $\zeta_x \in (a, b)$ such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\zeta_x)}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

2 Exercises:

Please submit solutions of problems with star(*) before 6:30PM on Wednesday and finish the rest by yourself.

1. * Let f be a function defined on $[a, b]$. Consider the following $n+1$ observation data:

x_0	x_1	$\cdots$	x_n
f_0	f_1	$\cdots$	f_n

(3)

where $x_0 = a$, $x_n = b$, $x_i \neq x_j$ for all $i \neq j$ and $f_i = f(x_i)$, $i = 0, 1, \cdots n$.

(a) Prove the existence and uniqueness of the polynomial interpolation $p_n(x)$ for the given data (3).

(b) Write down the basis functions $\{l_i(x)\}_{i=0}^n$ of Lagrange interpolation for the given data (3)

(c) Show that the basis functions $\{l_i(x)\}_{i=0}^n$ stated in (b) are linearly independent.

(d) Show that

$$\sum_{i=0}^{n} l_i(x) = 1.$$

(e) Write down the basis function of Newton's interpolation for the given data (3).

(f) Given the data (3), we define the divided difference recursively as follows:

$$f[x_i] := f(x_i), \quad f[x_0, x_1, ..., x_k] := \frac{f[x_{1,...,}x_k] - f[x_0, ..., x_{k-1}]}{x_k - x_0}$$

i. Let $i_0, i_1, ..., i_n$ be a rearrangement of the integers $0, 1, ..., n$. Show that

$$f[x_{i_0}, x_{i_1}, ..., x_{i_n}] = f[x_{0,}x_1, ..., x_n].$$

ii. Assume $x \neq x_i$, for $0 \leq i \leq n$,

$$f[x_0, ..., x_n, x] = \sum_{i=0}^{n} \frac{f[x, x_i]}{\Pi_{j=0, j \neq i}^{n}(x_i - x_j)}.$$

Solution. (a) As x_i are distinct point, the lagrange basis functions are well-defined. Therefore, the polynomial interpolation exists. Let p_1 and p_2 be two polynomial interpolation, and set $q(x) = p_1(x) - p_2(x)$. It is easy to see that $q(x_i) = 0$ for all $0 \leq i \leq n$. So q is a polynomial with degree $\leq n$ vanish at $n+1$ distinct point and thus $q = 0$, using the fundamental theorem of algebra.

(b) Basis function for Lagrange polynomials interpolation:

$$\Pi_{j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \quad i = 1, 2, \cdots, n.$$

(c) Let $\{\alpha_i\}_{i=0}^{n}$ a coefficients such that

$$\sum_{i=1}^{n} a_i l_i(x) = 0.$$

Taking $x = x_i$ in the equation above yields

$$\alpha_i = \alpha_i l_i(x_i) = \sum_{i=1}^{n} a_i l_i(x) = 0,$$

in view of the identity $l_i(x_j) = \delta_{ij}$.

(d) For any $x_1, ..., x_n$, the data are perfectly interpolated by the zeroth-order polynomial $P(x) = f(x) = 1$. Since the interpolation polynomial is unique, we have

$$1 = P(x) = \sum_{k=1}^{n} L_k(x)$$

for any x.

(e) Basis function for the Newton's polynomials interpolation:

$$1, x - x_0, (x - x_0)(x - x_1), \cdots \Pi_{i=0}^{n}(x - x_i).$$

(f) i. Let f_c and f_d be two polynomials, such that f_c interpolates f at $x_0, x_1, ..., x_n$ and f_d interpolates f at $x_{i_0}, x_{i_1}, ..., x_{i_n}$:

$$\begin{aligned} f_c &= c_0 + c_1(x - x_0) + ... + c_n(x - x_0)(x - x_1)...(x - x_{n-1}) \\ f_d &= d_0 + d_1(x - x_{i_0}) + ... + d_n(x - x_{i_0})(x - x_{i_1})...(x - x_{i_{n-1}}), \end{aligned}$$

We can rewrite the polynomials above as

$$\begin{aligned} f_c &= c_n x^n + ... \\ f_d &= d_n x^n + ... \end{aligned}$$

Since f_c and f_d were defined to be in the form of Newton's polynomials, we know that c_n and d_n are nth divided differences, $c_n = f[x_0, x_1, ..., x_n]$ and $d_n = f[x_{i_0}, ..., x_{i_n}]$. We also know that the polynomial interpolating the same nodes is unique. Thus the result follows.

ii. Let $\omega_{n+1} = \prod_{i=0}^{n}(x - x_i)$, we have

$$\begin{aligned} \sum_{i=0}^{n} l_i(x) &= \sum_{i=0}^{n} \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)} \\ \Rightarrow \frac{1}{\omega_{n+1}(x)} &= \sum_{i=0}^{n} \frac{1}{(x - x_i)\omega'_{n+1}(x_i)} \end{aligned}$$

We also have

$$\begin{aligned} f[x_0, ..., x_n, x] &= \frac{f(x) - p_n(x)}{\omega_{n+1}(x)} \\ p_n(x) &= \sum_{i=0}^{n} \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)} f(x_i) \end{aligned}$$

Then we have

$$\begin{aligned}
f[x_0, ..., x_n, x] &= \frac{f(x) - p_n(x)}{\omega_{n+1}(x)} \\
&= \frac{f(x)}{\omega_{n+1}(x)} - \frac{p_n(x)}{\omega_{n+1}(x)} \\
&= \frac{f(x)}{\omega_{n+1}(x)} - \sum_{i=0}^{n} \frac{f(x_i)}{(x - x_i)\omega'_{n+1}(x_i)} \\
&= \sum_{i=0}^{n} \frac{f(x)}{(x - x_i)\omega'_{n+1}(x_i)} - \sum_{i=0}^{n} \frac{f(x_i)}{(x - x_i)\omega'_{n+1}(x_i)} \\
&= \sum_{i=0}^{n} \frac{f(x) - f(x_i)}{(x - x_i)\omega'_{n+1}(x_i)} \\
&= \sum_{i=0}^{n} \frac{f[x, x_i]}{\omega'_{n+1}(x_i)}
\end{aligned}$$

□

2. * Consider the data

x	1	3/2	0
$f(x)$	3	13/4	3

(4)

(a) What are the Vandermonde interpolation polynomial, Langrange interpolation polynomial and Newton interpolation for these data?

(b) When we add one point to the data,

x	1	$\frac{3}{2}$	0	2
$f(x)$	3	$\frac{13}{4}$	3	$\frac{5}{3}$

(5)

What is the Newton interpolation now?

(c) Compute the Newton interpolation of the following data

x	0	1	2	3
$f(x)$	0	-5/2	-2	27/2

(6)

Evaluate the minimum of $f(x)$ over $[0, 3]$ based on the result above.

Solution. (a)

$$p(x) = 3 - \frac{1}{3}x + \frac{1}{3}x^2$$

$$L(x) = -6\left(x - \frac{3}{2}\right)x + \frac{13}{2}(x-1)x + 2(x-1)\left(x - \frac{3}{2}\right)$$

$$N(x) = 3 + \frac{1}{2}(x-1) + \frac{1}{3}(x-1)\left(x - \frac{3}{2}\right)$$

(b)

$$N(x) = 3 + \frac{1}{2}(x-1) + \frac{1}{3}(x-1)\left(x - \frac{3}{2}\right) - 2x(x-1)\left(x - \frac{3}{2}\right)$$

(c) Divided difference table:

$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$
-2.5	0.5	15.5	
1.5	7.5		
2			

Therefore

$$
\begin{aligned}
p(x) &= -2.5x + 1.5x(x-1) + 2x(x-1)(x-2) \\
&= 2x^3 - 4.5x^2
\end{aligned}
$$

And

$$p'(x) = 6x^2 - 9x$$

whose solutions are $x = 0$ and $x = 1.5$. Comparing the three values $p(0) = 0, p(1.5) = -\frac{27}{8}$ and $p(3) = \frac{27}{2}$, we know that the approximate minimum value of $f(x)$ is $-\frac{27}{8}$.

□

3. *

(a) Write three drawbacks for using Vandermonde interpolation.

(b) Consider the matrix

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

In this question, we try to prove the Vandermonde formula :

$$det(A) = \prod_{i>j}(x_i - x_j)$$

i. Show that it is true when $n = 1$.

ii. Conisder

$$f(t) = \det A = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n \\ 1 & t & t^2 & \cdots & t^n \end{pmatrix}$$

Show that

$$f(t) = k(t - x_0)\cdots(t - x_{n-1})$$

for some k and hence prove the Vandermonde formula.

Solution. (a) Finding inverse of matrix requires lots of calculation.
The matrix is ill-posed
Adding new data has to solve the linear system from the beginning.

(b) i. If $n=1$, $A=\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix}$, then $det(A)=X_1-X_0$. So $n=1$ is true.

ii. Consider

$$f(t)=detA=\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n \\ 1 & t & t^2 & \cdots & t^n \end{pmatrix}$$

Note that we can represent $f(t)$ as

$$f(t)=\pm D_0 \mp D_1 t \pm \cdots + D_n t^n$$

where D_i are determinants of $n\times n$ matrices that contain no factor of t. Since D_n is the Vandermonde determinant of the $n\times n$ matrix with coefficients x_0 through x_{n-1}, we have $f(t)$ an n^{th} degree polynomial with leading coefficients

$$k=\prod_{n>i>j}(x_i-x_j)$$

Moreover, if $t=x_0$, then $f(t)=f(x_0)=0$ and similar results can be obtained if $t=x_i, \quad i=1,2,...,n-1$. That is

$$f(x_0)=f(x_1)=...=f(x_{n-1})$$

Since the n values x_i, for $0\le i\le n$ are all distinct, and $f(t)$ is an n^{th} degree polynomial, we have

$$f(t)=k(t-x_0)\cdots(t-x_{n-1})$$

If we put $t=x_n$, we will have the Vandermonde formula. By the principle of M.I., we have proved the Vandermonde formula.

□

4. Let $x_0,x_1,...,x_n$ be distinct points and $l_j(x)$ be the Lagrange basis functions, prove the follow equality

$$\sum_{j=0}^{n}(x_j-x)^k l_j(x)\equiv 0, \quad k=1,2,...,n$$

Solution. Note that for any polynomial $p_n(x)$ of degree $\le n$, we have the

$$\sum_{i=0}^{n}p_n(x_i)l_i(x)=p_n(x)$$

Note that $(x_j-x)^k$ can also be written as a combination of $p_n(x)$. The result follows. □