MATH3230A Numerical Analysis

Tutorial 4 with solution

1 Recall:

1. **Symmetric positive definite matrix (SPD matrix):**
   Some useful properties of a SPD matrix are:

   (a) A SPD matrix is nonsingular.

   (b) Any diagonal square submatrix of an SPD matrix is also a SPD matrix.

   (c) Any eigenvalues of a SPD matrix is positive.

   (d) For any rectangular matrix U, if its column vectors are linearly independent, then the matrix U^TU is a SPD matrix.

   To check whether a symmetric matrix is positive definite or not, we have several ways:

   (a) The Sylvester's criterion states that a real-symmetric matrix A is positive definite if and only if all the leading principal minors of A are positive.

   (b) The eigenvalues of the matrix A are all positive.

   (c) Use the Cholesky Factorization to check (Matlab).

2. **Computational Complexity**
   A good indication on whether a particular numerical method is expensive is the computational complexity. All numerical algorithms can be decomposed into the basic components of vector-vector, matrix-vector and matrix-matrix operations, which all involve the basic operations (floating-point operations aka "flop") of addition, subtraction, multiplication and division of two numbers (floating points).

3. **Cholesky factorization:**
   Let us write

$$A = \begin{pmatrix} \alpha & a^T \\ a & A_{11} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & r^T \\ 0 & U_{11} \end{pmatrix}$$

   Then the Cholesky factorization runs as follows:

   (a) $\alpha = u_{11}^2$.

   (b) $a^T = u_{11}r^T$.

   (c) $A_{11} = rr^T + U_{11}^TU_{11}$.

   Or equivalently, we can write

   (a) $u_{11} = \sqrt{\alpha}$. (Take only the positive one)

   (b) $r^T = a^T/u_{11}$.

   (c) $U_{11}^TU_{11} = A_{11} - rr^T = \hat{A}_{11}$.

   One can repeat the above procedure for the submatrix $\hat{A}_{11}$. So the Cholesky factorization proceeds in n steps.

4. **LU factorization:**
   The Guassian elimination is basically a process of the so-called LU factorization for the matrix A. More preciously, if a matrix A can be written into $A = LU$, where the matrix L is a $n \times n$ lower triangular matrix with 1 as its diagonal entries, and the matrix U is an $n \times n$ upper triangular matrix. Then we say that A admits a LU factorization.

5. LDU **factorization:**
Suppose we have obtained an LU factorization of A:

$$A = \tilde{L}\tilde{U}.$$

Let $D = diag(\tilde{U})$. Then we can further factorize A as $A = LDU$, where L and U are lower and upper triangular matrices respectively, both matrices with 1 as their diagonal entries, and D is a diagonal matrix. For symmetric positive definite matrix A, the Cholesky factorization of A is $A = LL^T$. Now suppose the unique LDU factorization of A is

$$A = \tilde{L}D\tilde{U},$$

we have $\tilde{L}^T = \tilde{U}$ and hence $A = \tilde{L}D\tilde{L}^T$. Note that all diagonal entries of D are positive, we can therefore write

$$D = D^{\frac{1}{2}}D^{\frac{1}{2}},$$

where $D^{\frac{1}{2}}$ is a diagonal matrix with the main diagonal entries $\sqrt{D_{ii}}$. Then we have

$$A = \tilde{L}D^{\frac{1}{2}}D^{\frac{1}{2}}\tilde{L}^T = \tilde{L}D^{\frac{1}{2}}(\tilde{L}D^{\frac{1}{2}})^T = LL^T.$$

2 Exercises:

Please submit solutions of problems with star(*) before 6:30PM on Wednesday and finish the rest by yourself.

1. (a) * Write down the definition of a symmetric positive definite matrix.
   (b) * For any real $m \times n$ matrix M with its column vectors being linearly independent, prove that M^TM is a symmetric positive definite matrix.
   (c) * Write down a criterion to determine whether a matrix A is SPD. Check whether the following matrix is SPD by this criterion.

$$\begin{pmatrix} 8 & 6 & 3 \\ 6 & 7 & 2 \\ 3 & 2 & 4 \end{pmatrix}$$

   (d) Suppose A is SPD, prove that A^{-1} is also SPD by using eigenvalues of A and A^{-1}.

Solution. (a) An $n \times n$ matrix A is said to be symmetric and positive definite if it satisfies

   i. A is symmetric.
   ii. $x^TAx > 0$ for all $x \neq 0$.

(b) Since $(M^TM)^T = M^TM$, it is symmetric.
For any non-zero vector x, Mx is also a non-zero vector since the column vectors of M are independent. Therefore

$$x^TM^TMx = (Mx)^T(Mx) > 0.$$

Therefore M^TM is a positive definite.

(c) One of the following:

   i. The Sylvester's criterion states that a real-symmetric matrix A is positive definite if and only if all the leading principal minors of A are positive.
   ii. The eigenvalues of the matrix A are all positive.
   iii. Use the Cholesky Factorization to check

Now we use (i) to check.
The first order leading principal minor is $D_1 = 8$. The second order leading principal minor is

$$D_2 = \begin{vmatrix} 8 & 6 \\ 6 & 7 \end{vmatrix} = 20$$

The third order leading principal minor is

$$D_3 = \begin{vmatrix} 8 & 6 & 3 \\ 6 & 7 & 2 \\ 3 & 2 & 4 \end{vmatrix} = 57$$

Therefore the matrix is SPD.

(d) Assume λ is an eigenvalue of A, x is the eigenvector corresponding to λ. Then we have $Ax = \lambda x$. Furthermore, we have

$$A^{-1}x = \lambda^{-1}x$$

Therefore if λ is an eigenvalue of A, λ^{-1} is an eigenvalue of A^{-1}. When $\lambda > 0$, we also have $\lambda^{-1} > 0$. Hence A^{-1} is also a SPD.

□

2. Let A be a $n \times n$ matrix.

(a) Write down the definition of the Cholesky factorization.

(b) Calculate the total computational complexity of Cholesky factorization for large n.

(c) * Consider a SPD matrix A given by

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}.$$

Compute the Cholesky factorization of this matrix A.

(d) * In the algorithm, we generate the matrix $\hat{A}_{11} = A_{11} - rr^T$ in each step. Prove that the new matrix $\hat{A}_{11}$ is also a SPD matrix.

(e) * Using the result of the Cholesky factorization to show that the inverse of a SPD matrix A is also a SPD matrix.

Solution. (a) If A is an SPD matrix, then A can be factorized as U^TU, where U is a upper triangular matrix. If, in addition, we require the diagonal entries of U to be positive, then the factorization is unique and is called the Cholesky factorization of A.

(b) Check lecture notes page 73 for solution.

(c) Update the first row and the submatrix at the right bottom corner:

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ * & 2 & -2 \\ * & -2 & 4 \end{pmatrix}$$

Update the second row and the submatrix at the right bottom corner:

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ * & 2 & -2 \\ * & -2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ * & \sqrt{2} & -\sqrt{2} \\ * & * & 2 \end{pmatrix}$$

Update the third row and the submatrix at the right bottom corner:

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ * & \sqrt{2} & -\sqrt{2} \\ * & * & 2 \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ * & \sqrt{2} & -\sqrt{2} \\ * & * & \sqrt{2} \end{pmatrix}$$

Hence, if we set

$$U = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

then we have

$$A = U^TU.$$

(d) Using the same notation, we want to prove $\hat{A}_{11} := A_{11} - aa^T/\alpha = U_{11}^T U_{11}$ is also symmetric positive definite if A is symmetric positive definite.
To show that the matrix $\hat{A}_{11}$ is indeed an SPD matrix, for $\forall x \neq 0$, $x^T \in \mathbb{R}^{n-1}$, we construct $[x_1, x]^T \in \mathbb{R}^n$. Then we have

$$\begin{aligned}[x_1, x]A[x_1, x]^T &= x_1^2\alpha + x_1 a^T x + x_1 x^T a + x^T A_{11} x \\ &= x_1^2\alpha + 2x_1(a^T x) + \frac{1}{\alpha}(a^T x)(a^T x) + x^T \hat{A}_{11} x\end{aligned}$$

Now we find x_1 such that $x_1^2\alpha + 2x_1(a^T x) + \frac{1}{\alpha}(a^T x)(a^T x) = 0$. Note that the above equation is a simple second order nonlinear equation. Also note that $4(a^T x)^2 - 4\alpha \cdot \frac{1}{\alpha}(a^T x)(a^T x) = 0$. Therefore x_1 exists. For such x_1, we have $[x_1, x]A[x_1, x]^T = x^T \hat{A}_{11} x$. Since $x \neq 0$, we have $[x_1, x]^T \neq 0$. As A is SPD, we have $x^T \hat{A}_{11} x \neq$ for all $x \neq 0$. Therefore $\hat{A}_{11}$ is also SPD

(e) We set

$$B = U^{-1}(U^{-1})^T.$$

For the result above we know that B is a SPD matrix and

$$\begin{aligned}AB &= U^T U U^{-1}(U^{-1})^T = I \\ BA &= U^{-1}(U^{-1})^T U^T U = I\end{aligned}$$

So $B = A^{-1}$

□

3. Let A be a $n \times n$ non-singular matrix.

(a) Write down the definition of an LU factorization of A.

(b) * Consider the following system of linear equation $A\mathbf{x} = b$:

$$\begin{cases} x + 2y + 3z = 15 \\ 2x + 5y + 8z = 37 \\ 3x + 4z = 10 \end{cases}$$

Find a LU factorization of A.

(c) * Is your result in (b) a unique LU factorization of A? If not, please give an example of another LU factorization of A.

(d) Write down the corresponding steps of Gaussian elimination and then solve the above system.

Solution. (a) If there exist an $n \times n$ lower triangular matrix L with 1 as its diagonal entries and an $n \times n$ upper matrix U such that

$$A = LU,$$

then we say that A admits a LU factorization.

(b) Let

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad \text{then } L_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -5 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix}, \quad \text{then } L_2 L_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{bmatrix} = U$$

Let

$$L = (L_2 L_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -6 & 1 \end{bmatrix}$$

Then

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

(c) Yes.

(d) The Gaussian elimination steps are the same as the steps that we do LU factorization in (b). First we solve $L\mathbf{y} = b$, we have

$$\begin{aligned} y_1 &= 15 \\ y_2 &= 7 \\ y_3 &= 7 \end{aligned}$$

Then we solve $U\mathbf{x} = \mathbf{y}$, we have

$$\begin{aligned} x &= 2 \\ y &= 5 \\ z &= 1 \end{aligned}$$

□