MATH3230A Numerical Analysis

Tutorial 1 with solution

1 Recall:

1. **Bisection method:**
   The bisection algorithm is based on the intermediate value theorem:

   Theorem 1 (Intermediate value theorem)**.** *Let $f(x)$ be continuous on $[a, b]$, then for any real number g that lies between $f(a)$ and $f(b)$ there exists $\zeta \in [a, b]$ such that $f(\zeta) = g$.*

   We then have the following

   Theorem 2. *If $f(a)f(b) < 0$, then there exists at least one solution $x^* \in (a, b)$ such that $f(x^*) = 0$.*

   For the algorithm, please read Page 14 of the Lecture notes.

2. **Convergence of Bisection Algorithm:**

   Theorem 3. *Let $f(x)$ be a continuous function on $[a, b]$ such that $f(a)f(b) < 0$, then the bisection algorithm always converges to a solution x^* of the equation $f(x) = 0$, and the following error estimate holds for the $k-th$ approximate value x_k:*

   $$|x_k - x^*| \leq \frac{1}{2}(b_k - a_k) = 2^{-(k+1)}(b - a)$$

3. **Newton's method:**
   Newton's method is an iterative method. Basically it begins with an initial guess x_0, and produces successive approximate values

   $$x_1, x_2, x_3, \ldots, x_k, \ldots$$

   The Newton's method is:

   $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots$$

   Assuming that $f(x)$ is continuously differentiable up to second order near $x = x^*$, we have the following results:

   Theorem 4 (Local convergence of Newton's method)**.** *As long as the initial guess x_0 is taken such that $|x_0 - x^*| \leq \delta$, then all x_k generated by the Newton's method lie in the range $|x - x^*| \leq \delta$.*

   Theorem 5 (Quadratic convergence of Newton's method)**.** *The Newton's method is a local convergence iterative method, and it has quadratical convergence when the initial guess x_0 is taken within the convergence region.*

2 Exercises:

Please do the star problem (*) in tutorial class first and finish the rest after class.

1. State the definition of the following concepts:

   (a) Linear convergence of a sequence;

   (b) Sublinear convergence of a sequence;

   (c) Superlinear convergence of a sequence;

   (d) Convergence of a sequence with order p;

   (e) Quadratic convergence of a sequence.

   Solution.

   (a) Linear convergence of a sequence: Assume that a sequence $\{x_k\}$ converges to x^*, it is said to converge linearly to x^* if

   $$\lim_{k\to\infty} \frac{|x_{k+1}-x^*|}{|x_k-x^*|} = \rho,$$

   where $0 < \rho < 1$ is a constant.

   (b) Sublinear convergence of a sequence: If $\rho = 1$, we say that $\{x_k\}$ coverge to x^* sublinearly.

   (c) Superlinear convergence of a sequence: If $\rho = 0$, we say that $\{x_k\}$ coverge to x^* superlinearly.

   (d) Assume that $p > 1$ and a sequence $\{x_k\}$ converges to x^*, it is said to converge to x^* with order p if there exists a positive constant $C > 0$ such that

   $$\lim_{k\to\infty} \frac{|x_{k+1}-x^*|}{|x_k-x^*|^p} = C.$$

   (e) Quadratic convergence of a sequence: If there is a positive constant C such that

   $$\lim_{k\to\infty} \frac{|x_{k+1}-x^*|}{|x_k-x^*|^2} = C,$$

   then the sequence $\{x_k\}$ is said to converge to x^* quadratically or the sequence is of quadratic convergence.

   □

2. Check whether the following sequences converge. In the case of convergence, find the rate of convergence or the convergence order.

   (a) * $x_0 = 1$, $x_{n+1} = \frac{1}{2}\ln\sqrt{x_n+1}$, $n = 0, 1, 2, \cdots$.

   (b) $x_0 = 3$, $x_n = 3 \times 2^{-n}$, $n = 0, 1, 2, \cdots$.

   (c) * $x_n = \sum_{k=1}^{n} \frac{1}{k}$, $n = 1, 2, \cdots$.

   (d) $x_0 = \frac{1}{2}$, $x_1 = 1$, $x_{n+1} = x_n + \frac{x_n^2}{x_n^2 + x_{n-1}^2}$, $n = 1, 2, \cdots$.

   (e) $x_1 = 2$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$, $n = 1, 2, \cdots$.

   (f) $x_0 > 0$, $x_{n+1} = (2^{-n} + 3^{-n})\, x_n$, $n = 0, 1, 2, \cdots$.

   (g) * $x_n = 1 + 2^{1-n} + \frac{1}{(n+2)^n}$, $n = 0, 1, 2, ...$

Solution.

(a) first we guess the limit point $x^\star$, which must satisfy

$$x^\star = \frac{1}{2} ln\sqrt{x^\star + 1},$$

so $x^\star$ is the solution of

$$f(x) = x - \frac{1}{4} ln(x+1) = 0.$$

It is easy to find that for $x \in (0,1]$, $f'(x) > 0$ and $f(1) > 0$, $f(0) = 0$. So we conclude that $f(0) = 0$, $x^\star = 0$. Then

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{\frac{1}{4} ln(x_n + 1)}{x_n} = \lim_{n\to\infty} \frac{1}{4(x_n+1)} = \frac{1}{4}$$

So this sequence converges to 0 linearly.

(b) $x_n = 3 \times 2^{-n}$, $n = 1, 2, ...$, then it is easy to compute that

$$\lim_{n\to\infty} x_n = 0.$$

Thus

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = 0.5.$$

Then the sequence converges to 0 linearly.

(c) $x_n = \sum_{k=1}^{n} \frac{1}{k}$, $n = 1, 2, \cdots$. It is easy to see that

$$\sum_{k=2}^{n} \frac{1}{k} \geq \sum_{k=2}^{n} \int_k^{k+1} \frac{1}{x} dx = \int_2^{n+1} \frac{1}{x} dx = \ln(n+1) - \ln(2) \to \infty$$

as $n \to \infty$, which implies that x_n diverges.

(d) $x_0 = \frac{1}{2}$, $x_1 = 1$, $x_{n+1} = x_n + \frac{x_n^2}{x_n^2 + x_{n-1}^2}$, $n = 1, 2, \cdots$. From the interation scheme it follows $x_n > x_{n-1}$. Thus

$$2x_n^2 \geq x_n^2 + x_{n-1}^2,$$

which implies that

$$\frac{x_n^2}{x_n^2 + x_{n-1}^2} \geq \frac{1}{2}.$$

So,

$$x_{n+1} = x_1 + \sum_{k=1}^{n} (x_{k+1} - x_k) = x_1 + \sum_{k=1}^{n} \frac{x_k^2}{x_k^2 + x_{k-1}^2} \geq x_1 + \frac{n}{2} \to \infty$$

as $n \to \infty$, thus the sequence $\{x_n\}$ diverges.

(e) $x_1 = 2$, $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{x_n})$, $n = 1, 2, \cdots$.
First, we need to guess the limit x^*, which needs to satisfy

$$x^* = \frac{1}{2}(x^* + \frac{1}{x^*}).$$

So, $x^* = 1$. To prove it, by MI, we can prove

$$x_n \geq 1 \quad \forall n \geq 1,$$

then we have

$$|x_{n+1} - 1| = \frac{(x_n - 1)^2}{2x_n} \leq (\frac{1}{2})(x_n - 1)^2 \leq (\frac{1}{2})(\frac{1}{2})^2 (x_{n-1} - 1)^4 \leq \frac{1}{2^{2^n - 1}} \to 0$$

as $n \to \infty$, thus the sequence $\{x_n\}$ converges to 1.

$$\lim_{n\to\infty} \frac{|x_{n+1} - 1|}{|x_n - 1|^2} = \frac{1}{2}.$$

The sequence $\{x_n\}$ converges to 1 quadratically.

(f) $x_0 > 0$, $x_{n+1} = (2^{-n} + 3^{-n})\, x_n$, $n = 0, 1, 2, \cdots$.
It is easy to see that

$$2^{-n} + 3^{-n} \le 1/2 + 1/3 \le \frac{5}{6}.$$

Thus

$$|x_n| \le (\frac{5}{6})^n x_0 \to 0$$

as $n \to \infty$.

$$\lim_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim_{n\to\infty} (2^{-n} + 3^{-n}) = 0.$$

the sequence $\{x_n\}$ converges to 0 superlinearly.

(g) $x_n = 1 + 2^{1-n} + \frac{1}{(n+2)^n}$, $n = 0, 1, 2, \dots$ Then it is clear that

$$\lim_{n\to\infty} x_n = 1.$$

Thus

$$\lim_{n\to\infty} |\frac{x_{n+1} - 1}{x_n - 1}| = \lim_{n\to\infty} \frac{2^{-n} + \frac{1}{(n+3)^{n+1}}}{2^{1-n} + \frac{1}{(n+2)^n}} = 0.5.$$

So the sequence converges linearly.

□

3. *

(a) Show that a sequence which converges with order $\alpha > 1$ must converge superlinearly.

(b) Show that the sequence $x_n = \frac{1}{n^n}$ converges superlinearly to 0 but does not converge to 0 with order α for any $\alpha > 1$.

Solution.

(a) Because the sequence $\{x_n\}$ converges to x^* with order $\alpha > 1$, we have

$$\lim_{k\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = C.$$

Then

$$\lim_{k\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = \lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} \cdot \lim_{n\to\infty} |x_n - x^*|^{\alpha-1} = C \cdot 0 = 0.$$

(b) Clearly, $x_n = \frac{1}{n^n} \to 0$ as $n \to \infty$. Then, we have

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{n} \big(1 + \frac{1}{n}\big)^{-(n+1)} = \lim_{n\to\infty} \frac{1}{n} \lim_{n\to\infty} \big(1 + \frac{1}{n}\big)^{-(n+1)} = 0 \cdot e^{-1} = 0.$$

Therefore, the sequence converges superlinearly to 0.
Then, for $\alpha > 1$, assume the sequence converges to 0 with order α, that is,

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^\alpha} = C.$$

But,

$$\infty = \lim_{n\to\infty} n^{(\alpha-1)n-1} = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^\alpha} \big(1 + \frac{1}{n}\big)^{n+1} = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^\alpha} \lim_{n\to\infty} \big(1 + \frac{1}{n}\big)^{n+1} = C \cdot e.$$

Then, we have $C = \infty$, which contradicting that C is constant.
Hence the sequence does not converge to 0 with order α for $\alpha > 1$.

□

4. (a) Apply Newton's method to the equation $\frac{1}{x} - a = 0$ to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - ax_n^2.$$

(b) Use part (a) to compute the approximation value of $1/2.14$ with initial guess $x_0 = \frac{1}{2}$ at the 3^{rd} iteration.

Solution.

(a) Consider the following non-linear equation:

$$f(x) := \frac{1}{x} - a,$$

$$f'(x) = -\frac{1}{x^2}.$$

Newton's method gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x^n} - a}{-\frac{1}{x_n^2}} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$$

(b) Let $a = 2.14$ in the above expression, so

$$\begin{aligned} x_0 &= 0.5 \\ x_1 &= 2x_0 - ax_0^2 = 0.465 \\ x_2 &= 2x_1 - ax_1^2 = 0.467278 \\ x_3 &= 2x_2 - ax_2^2 = 0.467289 \end{aligned}$$

□

5. (a) * Apply Newton's method to find $\sqrt[m]{a}$ with $a > 0$ and m is a positive integer.

(b) * For $m > 1$, show that the convergence order of the sequence is 2.

(c) Use part (a) to compute the approximation value of $\sqrt[3]{2}$ with initial guess $x_0 = 1$ at the 3^{rd} iteration.

Solution.

(a) Consider the following non-linear equation:

$$f(x) := x^m - a,$$

$$f'(x) = mx^{m-1}.$$

Newton's method gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^m - a}{mx_n^{m-1}} = (1 - \frac{1}{m})x_n + \frac{a}{mx_n^{m-1}} = \frac{(m-1)x_n^m + a}{mx_n^{m-1}}$$

(b) Let $e_k = x_k - x^\star$, then

$$\begin{aligned} \lim_{k\to\infty} |\frac{e_{k+1}}{e_k^2}| &= \lim_{k\to\infty} |\frac{\frac{m-1}{m}x_k + \frac{a}{m}x_k^{1-m} - a^{\frac{1}{m}}}{(x_k - a^{\frac{1}{m}})^2}| \\ &= \lim_{k\to\infty} |\frac{\frac{m-1}{m} + \frac{a(1-m)}{m}x_k^{-m}}{2(x_k - a^{\frac{1}{m}})}| \quad (l'Hopital'sRule) \\ &= \lim_{k\to\infty} |\frac{a(m-1)x_k^{-m-1}}{2}| \quad (l'Hopital'sRule) \\ &= \frac{(m-1)a^{\frac{-1}{m}}}{2} \end{aligned} \tag{1}$$

(c) Let $a = 2$ and $m = 3$ in the above expression, so

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1.333333 \\ x_2 &= 1.263889 \\ x_3 &= 1.259933 \end{aligned}$$

$\square$

6. Consider a real nonlinear function f, and the nonlinear equation:

$$f(x) = 0.$$

(a) Prove the local convergence of the Newton's method applied to f to find the solution x^*.

(b) Conclude the assumptions that you made in your proof in (a).

(c) * The choice of initial guess x_0 is very important to the convergence of the Newton's method. Try the Newton's method for the following special function f. Let the function $f : [0, 10] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2, & \text{if } x \in [0, 2), \\ 6 - x, & \text{if } x \in [2, 3), \\ 3, & \text{if } x \in [3, 10]. \end{cases} \tag{2}$$

i. We first choose an initial guess $0 < x_0 < 2$. Does the Newton's method work? Please prove your claim.

ii. Now we choose an initial guess $2 < x_0 < 3$. Does the Newton's method work? Please explain why briefly.

(d) Derive an iterative method to solve the nonlinear equation (2) based on the Taylor's expansion up to the second order approximation.

Solution. (a) Define $\varphi(x) = x - \frac{f(x)}{f'(x)}$, then the Newton's method can be written as the fixed-point iteration:

$$x_{k+1} = \varphi(x_k), \quad k = 0, 1, 2, ...$$

By direct substitution, we have

$$x^* = \varphi(x^*)$$

Let $e_k = x_k - x^*$ be the error at the k-th iteration, then

$$e_{k+1} = \varphi(x_k) - \varphi(x^*)$$

By the mean-value theorem, we can have the following error equation for e_k:

$$e_{k+1} = \varphi'(\zeta_k)(x_k - x*) = \varphi'(\zeta_k)e_k,$$

where ζ_k lies between x_k and x^*. Since

$$\varphi'(x) = 1 - \frac{f'(x)f'(x) - f''(x)f(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2},$$

and $f(x^*) = 0$, we have

$$\varphi'(x^*) = \frac{f(x^*)f''(x^*)}{f'(x^*)^2} = 0$$

Therefore, we can find a constant $\delta > 0$ such that,

$$|\varphi'(x)| \leq \frac{1}{2}, \quad x \in \{x \mid |x - x^*| \leq \delta\}$$

In fact, by induction it is easy to check that for $k = 1, 2, ...,$ we have

$$\varphi'(\zeta_k) \le \frac{1}{2} < 1, \quad |e_{k+1}| \le \frac{1}{2}|e_k| \le \delta$$

This yields

$$|e_{k+1}| \le \frac{1}{2^{k+1}}|e_0| = \frac{1}{2^{k+1}}|x_0 - x^*|,$$

so we have $e_k \to 0$ as $k \to \infty$, namely $x_k \to x^*$ as $k \to \infty$. This demonstrates the convergence of the Newton's method.

(b) i. $f(x^*) = 0$

ii. f is continuously differentiable up to 2nd order.

iii. $f'(x^*) \neq 0$

(c) i. As $x_0 \in (0, 2)$, $x_{n+1} = x_n - \frac{x_n^2}{2x_n} = \frac{x_n}{2}$, which implies that

$$x_n = \frac{1}{2^n}x_0$$

Thus the Newton's method works.

ii. As $x_0 \in (2, 3)$, $x_1 = 6$. Iteration stops as $f'(x_1) = 0$.

(d) Given an initial guess x_0, the Taylor expansion yields

$$F(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 \approx f(x).$$

So instead of solving $f(x) = 0$, we can solve $F(x) = 0$. Solving the quadratic equation, we have

$$x = x_1 = x_0 + \frac{-f'(x_0) \pm \sqrt{(f'(x_0))^2 - 2f(x_0)f''(x_0)}}{f''(x_0)}$$

Obviously we need to assume the initial x_0 is close enough such that $2f(x_0)f''(x_0) \le (f'(x_0))^2$ is guaraneted. For the issue of the choice of the sign, we should indeed take the one which leaves us closest to our current iteration. That is, we should choose the root such that $x_{n+1} - x_n$ is smaller in magnitude. There are two reasons. The first is that another root may be far away from current x_n or the exact solution. The second one is that we hope to prevent our scheme to bounce back and forth during iterations. In order to minimize the increment size we can choose the root such that it is opposite to $-f(x_0)'$. Therefore we have

$$x_{n+1} = x_n - \frac{f'(x_n) - sign(f'(x_n))\sqrt{(f'(x_n))^2 - 2f(x_n)f''(x_n)}}{f''(x_n)}.$$

Assume that $f(x)$ satisfies the following conditions:
$f(x) = 0$ has a solution x^*.
f is continuously differentiable up to third order.
$f'(x^*) \neq 0$ and $f''(x^*) \neq 0$. If we set

$$\phi(x) = x + \frac{-f'(x) + sign(f'(x))\sqrt{(f'(x))^2 - 2f(x)f''(x)}}{f''(x)}$$

then the function ϕ is continuously differentiable near x^* and

$$\phi'(x^*) = 1 + \frac{1}{(f''(x))^2}\left[(f''(x))^2\left(\frac{dsign(f'(x))}{dx}|f'(x)| - 1\right)\right] = 0$$

Hence, there exists a neighborhood $B_\delta(x^*)$ of x^* such that $\phi'(x) \le \frac{1}{2}$. Thus, the sequence $\{x_n\}$ locally converges to x^*.

$\square$

7. Consider the Newton's method for solving the following nonlinear equation:

$$\frac{1}{x^3} = -a,$$

where $a > 0$. Let $f(x) = \frac{1}{x^3} + a$ and assume that the initial guess $x_0 \in \left(-a^{-\frac{1}{3}}, 0\right)$.

(a) * Write down the iterative formula resulting from the Newton's method for solving $f(x) = 0$.

(b) * Show that the sequence in (a) converges to $-a^{-\frac{1}{3}}$.

(c) Show that the convergence order of the sequence is two.

Solution. (a) Since $f'(x) = -3x^{-4}$. Newton's method gives:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^{-3} + a}{-3x_k^{-4}} = \frac{4}{3}x_k + \frac{a}{3}x_k^4$$

(b) First we show that $-a^{-\frac{1}{3}} < x_k < 0, \forall k$. Note that $k = 0$ is valid by assumption. Assume $-a^{-\frac{1}{3}} < x_k < 0$ hold, consider a function $g(x) = \frac{4x}{3} + \frac{ax^4}{3}$, $x \in (-a^{-\frac{1}{3}}, 0)$, $g'(x) = \frac{4}{3} + \frac{4ax^3}{3} > 0$ and so $g(x)$ is monotone, it's maximum and minimum are obtained at interval end, $g(0) = 0, g(-a^{-\frac{1}{3}}) = -a^{-\frac{1}{3}}$. Finally we have $-a^{-\frac{1}{3}} < x_{k+1} < 0$. On the other hand,

$$\frac{x_{k+1} - x_k}{x_k} = \frac{1 + ax_k^3}{3}$$

since $-a^{-\frac{1}{3}} < x_k$, we have

$$0 < \frac{1 + ax_k^3}{3} < \frac{1}{3}$$

since $x_k < 0$, we have

$$x_{k+1} - x_k < 0$$

so the sequence is decreasing. We conclude that the sequence is convergent. Taking limit on both sides,

$$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \frac{4}{3}x_k + \frac{a}{3}x_k^4$$

$$x^* = \frac{4}{3}x^* + \frac{a}{3}x^{*4}$$

We get $x^* = -a^{-\frac{1}{3}}$.

(c) Let $e_k = x_k - x^*$ By (b), we have

$$\lim_{k\to\infty} x_k = -a^{-\frac{1}{3}}$$

Then

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{e_{k+1}}{e_k^2}\right| &= \lim_{k\to\infty} \left|\frac{\frac{4}{3}x_k + \frac{a}{3}x_k^4 + a^{-\frac{1}{3}}}{(x_k + a^{-\frac{1}{3}})^2}\right| \\
&= \lim_{k\to\infty} \left|\frac{\frac{4}{3} + \frac{4}{3}ax_k^3}{2(x_k + a^{-\frac{1}{3}})}\right| \quad \text{(L' Hopital's rule)} \\
&= \lim_{k\to\infty} \left|\frac{4ax_k^2}{2}\right| \quad \text{(L' Hopital's rule)} \\
&= \lim_{k\to\infty} |2ax_k^2| \\
&= a^{\frac{1}{3}}
\end{aligned}$$

Hence the convergence order of the sequence is 2.

□