MATH3230A - Numerical Analysis
Exercises on Numerical Integration

1. Derive the Newton–Cotes rule for $\int_0^1 f(x)\,dx$ based on the points 0, 1/3, 2/3 and 1.

2. Obtain a Newton–Cotes rule for $\int_0^1 f(x)\,dx$ that is exact for all polynomials of degree ≤ 4.

3. Find the coefficients a_0, a_1 such that

$$\int_0^1 f(x)\,dx \approx a_0 f(0) + a_1 f(1)$$

   is exact for functions of the form $f(x) = ae^x + b\cos(\pi x/2)$.

4. Find the coefficients a_0, a_1 such that

$$\int_0^{2\pi} f(x)\,dx = a_0 f(0) + a_1 f(\pi)$$

   is exact for all functions of the form $f(x) = a + b\cos x$.

5. Derive a formula for approximating

$$\int_1^3 f(x)\,dx$$

   in terms of $f(0)$, $f(2)$ and $f(4)$ which is exact for all polynomials of degree ≤ 2.

6. For which class of polynomials is the quadrature rule

$$\int_0^2 x f(x)\,dx \approx a_0 f(0) + a_1 f(1) + a_2 f(2)$$

   exact?

7. Write down a composite rectangular rule based on

$$\int_0^1 f(x)\,dx \approx f(1).$$

8. Compute the error estimate for the following Newton-Cotes rules:

(a) $\int_0^1 f(x)\,dx \approx af(0) + bf(1/2) + cf(1)$;

(b) $\int_0^1 f(x)\,dx \approx \alpha f(1/4) + \beta f(1/2) + \gamma f(3/4)$;

9. Examine whether there is a quadrature rule of the form

$$\int_0^1 f(x)\,dx \approx a\Big(f(x_0) + f(x_1)\Big)$$

for some a, x_0, x_1 that is exact for all polynomials of degree ≤ 2.

10. Determine the number of subintervals needed in order to approximate

$$\int_1^2 f(x)\,dx$$

to an accuracy of 10^{-6} using the composite trapezoidal rule for

(a) $f(x) = x$;

(b) $f(x) = e^{-x}$;

(c) $f(x) = e^{-x^2}$.

11. Let $w : [a,b] \to (0,\infty)$ be a weight, and $f : [a,b] \to \mathbb{R}$. Given an equally-spaced partition of $[a,b]$ into n subintervals with points $a = x_0 < x_1 < \cdots < x_n = b$, identify the coefficients $\alpha_0, \ldots, \alpha_n$ for the quadrature rule

$$\int_a^b w(x)f(x)\,dx \approx \sum_{i=0}^{n} \alpha_i f(x_i)$$

so that it is exact for all polynomials of degree $\le n$.

12. Derive the Gauss–Legendre quadrature rule with 4 nodal points in $[-1,1]$.

13. Find a Gauss–Legendre quadrature rule for

$$\int_{-1}^1 xf(x)\,dx \approx a_0 f(x_0) + a_1 f(x_1)$$

that is exact for all polynomials of degree ≤ 3.

14. Repeat the above but now with $x^2 f(x)$ as the integrand.