# MATH3230A - Numerical Analysis
# Exercises on Nonlinear systems of equations

## 1 Newton's method

1. Perform two Newton iteration for the system
$$xy^2 + x^2y + x^4 = 3, \quad x^4y^5 - 2x^5y - x^2 = -2$$
starting with $(1,1)$.

2. Perform two Newton iteration for the system
$$xy = z^2 + 1, \quad xyz + y^2 = x^2 + 2, \quad e^x + z = e^y + 3$$
starting with $(1,1,1)$.

## 2 Broyden's method

1. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix with vectors $u, v \in \mathbb{R}^n$. Show that
$$\frac{\left(\frac{A^{-1}u}{1+v\cdot A^{-1}u} - A^{-1}u\right) \otimes A^{-T}v}{v \cdot A^{-1}u} = \frac{A^{-1}(u \otimes v)A^{-1}}{1 + v \cdot A^{-1}u},$$
where $A^{-T} = (A^{-1})^T = (A^T)^{-1}$.

2. Given a matrix $C \in \mathbb{R}^{n\times n}$ and vectors $w, z, g \in \mathbb{R}^n$, the matrix $D \in \mathbb{R}^{n\times n}$ satisfying $Dw = z$ and $Dy = Cy$ for any $y \cdot g = 0$ has the form
$$D = C - \frac{(z - Cw) \otimes g}{g \cdot w}.$$
Use the above formula, as well as the conditions
$$A_k(x_k - x_{k-1}) = F(x_k) - F(x_{k-1}), \quad A_k y = A_{k-1}y \text{ for } y \cdot (x_k - x_{k-1}) = 0,$$
show that
$$A_k = A_{k-1} + \frac{(F(x_k) - F(x_{k-1}) - A_{k-1}d_{k-1}) \otimes d_{k-1}}{d_{k-1} \cdot d_{k-1}}, \quad d_{k-1} = x_k - x_{k-1}.$$
Use the above formula to derive an alternative variant of Broyden's method using $A_k$ instead of $A_k^{-1}$.

3. Perform two iterations of the good and bad Broyden method to
$$xy^2 + x^2y + x^4 = 3, \quad x^4y^5 - 2x^5y - x^2 = -2$$
starting with $(x_0, y_0) = (1,1)$ and $A_0 = DF(x_0, y_0)$.

# 3 Steepest descent

1. Compute the Hessian for the function $g(x) = \frac{1}{2}x^T Ax - b^T x$ where $A$ is a symmetric matrix.

2. Let $A$ be a symmetric matrix with $Ax = b$. Let $y$ be any vector. Show that

$$\frac{1}{2}(x-y)^T A(x-y) = \frac{1}{2}b^T A^{-1}b + g(y).$$

   Explain why minimizing $g(y)$ is equivalent to minimizing $\frac{1}{2}(x-y)^T A(x-y)$.

3. For $x$ such that $Ax = b$, compute the gradient of $g(y) = \frac{1}{2}(x-y)^T A(x-y)$.

4. Let $A$ be a positive definite and symmetric matrix. Recall that the steepest descent method for solving $Ax = b$ is: Select $x_0$, for $k = 0, 1, 2, \ldots$ do the following:

   (i) Compute $d_k = b - Ax_k$ and $\alpha_k = \frac{d_k \cdot d_k}{d_k \cdot Ad_k}$.

   (ii) Update $x_{k+1} = x_k - \alpha_k d_k$.

   Suppose at some iteration $k$, the error vector $e_k = x_k - x^*$ is an eigenvector of $A$, show that

   (a) $d_k = -\lambda e_k$ where $\lambda$ is the corresponding eigenvalue of $e_k$;

   (b) $e_{k+1} = 0$.

   This shows that the subsequent descent step moves directly to the correct solution to $Ax = b$.

5. Perform two iterations of the steepest descent method for the system

$$\begin{pmatrix} 2 & 0 & -1 \\ -2 & -10 & 0 \\ -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -12 \\ 2 \end{pmatrix}$$

   with starting vector $(0, 0, 0)$.

6. Let $A$ be an SPD matrix with eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

   Consider the sequence $\{x_k\}$ generated by

$$x_{k+1} = x_k - \alpha(b - Ax_k),$$

   where $x^*$ is the solution to $Ax = b$. Show that

   (i) $x_{k+1} - x^* = (I - \alpha A)(x_k - x^*)$;

   (ii) The eigenvalues of the matrix $I - \alpha A$ are $\{1 - \alpha\lambda_i\}_{i=1}^n$;

   (iii) The sequence $\{x_k\}$ converges to $x^*$ if $\alpha < \frac{2}{\lambda_n}$.