MATH1010 University Mathematics
L'Hopital's rule and Taylor series exercies

1. Use L'Hopital's rule to evaluate the following limits.

(a) $\lim_{x\to 0} \dfrac{e^x - 1 - \sin x}{x \sin x}$

(b) $\lim_{x\to 0} \dfrac{2\sin x - \sin 2x}{x - \sin x}$

(c) $\lim_{x\to 0} \dfrac{\ln(1-x^2)}{\ln \cos x}$

(d) $\lim_{x\to 0} \dfrac{\sin^{-1} x}{x}$

(e) $\lim_{x\to 0} \dfrac{\tan^{-1} x}{1 - \sqrt{1-x}}$

(f) $\lim_{x\to 0} x \ln(1 - \cos x)$

(g) $\lim_{x\to 0} \sin x \ln x$

(h) $\lim_{x\to 0^+} x^3 (\ln x)^2$

(i) $\lim_{x\to 0^+} x \ln(1 + e^{\frac{1}{x}})$

(j) $\lim_{x\to 0^-} x \ln(1 + e^{\frac{1}{x}})$

(k) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right)$

(l) $\lim_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x - 1} \right)$

(m) $\lim_{x\to 0} \left(\dfrac{1}{\ln(1+x)} - \dfrac{1}{\sin x} \right)$

(n) $\lim_{x\to 0} \dfrac{2^x - 1}{\tan 2x}$

(o) $\lim_{x\to 0} \dfrac{3^x - 1}{2^x - 1}$

2. Use L'Hopital's rule to evaluate the following limits.

(a) $\lim_{x\to +\infty} \dfrac{(\ln x)^2}{x}$

(b) $\lim_{x\to +\infty} \dfrac{\ln(1+x^4)}{\ln(1+x)}$

(c) $\lim_{x\to +\infty} \dfrac{\ln(2x^3 - 5x^2 + 3)}{\ln(4x^2 + x - 7)}$

(d) $\lim_{x\to +\infty} \dfrac{\ln(e^{4x} + x^3)}{\ln(e^x + x)}$

(e) $\lim_{x\to +\infty} x(2^{\frac{1}{x}} - 1)$

(f) $\lim_{x\to +\infty} x \sin^{-1}\left(\dfrac{1}{x}\right)$

(g) $\lim_{x\to +\infty} x \left(\dfrac{\pi}{2} - \tan^{-1} x \right)$

(h) $\lim_{x\to +\infty} x \ln\left(1 + \dfrac{3}{x}\right)$

3. Evaluate the following limits.

(a) $\lim_{x\to 0^+} x^{x^2}$

(b) $\lim_{x\to 0^+} x^{\frac{1}{1+\ln x}}$

(c) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

(d) $\lim_{x\to 0^+} (\sin x)^x$

(e) $\lim_{x\to 0^+} (\sin x)^{\sin x}$

(f) $\lim_{x\to 0} (1+\sin x)^{\dfrac{1}{\tan x}}$

(g) $\lim_{x\to 0} \dfrac{(e+x)^{1/x}}{e^{1/x}}$

(h) $\lim_{x\to 0} \dfrac{(1+x)^x - 1}{x^2}$

(i) $\lim_{x\to 0} \dfrac{e-(1+x)^{\frac{1}{x}}}{x}$

(j) $\lim_{x\to +\infty} (1+5e^{2x})^{\frac{3}{x}}$

(k) $\lim_{x\to +\infty} (1+3x)^{\frac{2}{\ln x}}$

(l) $\lim_{x\to +\infty} \dfrac{x+\sin(e^x)}{x+e^{\sin x}}$

(m) $\lim_{x\to +\infty} \left(\dfrac{e^x+\cos x}{x^2+1}\right)^{\frac{1}{x}}$

4. For the following functions $f(x)$, find $f^{(k)}(0)$ for $k = 0, 1, 2, 3, 4$ and write down the Taylor polynomial of $f(x)$ of degree 3 at $x = 0$.

(a) $f(x) = (1+x)\ln(1+x)$

(b) $f(x) = e^{\sin x}$

(c) $f(x) = \sec x$

(d) $f(x) = \tan x$

5. Find the Taylor series of the following functions up to the term in x^3 at $x = 0$.

(a) $\dfrac{1}{(1+x)^2}$

(b) $\sqrt{1-3x}$

(c) $\dfrac{x}{(1-2x)^3}$

(d) $\dfrac{1}{x+3}$

(e) $\dfrac{x}{(x-1)(x-2)}$

(f) e^{x+2}

(g) $\ln(e+x)$

(h) $\cos^2 x$

(i) $\sin x \cos 2x$

(j) $e^{2x}\cos x$

(k) $(1+\sin x)^2$

(l) 3^x

(m) $\sin^{-1} x$

(n) $\ln(x+\sqrt{1+x^2})$

6. Find the Taylor series up to the term in $(x-c)^3$ of the functions at $x = c$.

(a) $\dfrac{1}{1+x}$; $c = 1$.

(b) $\dfrac{2-x}{3+x}$; $c = 1$.

(c) $\sqrt{x}$; $c = 4$.

(d) $\sqrt[3]{5+3x}$; $c = 1$.

(e) e^x; $c = 1$.

(f) $\ln x$; $c = 1$.

(g) $\sin x$; $c = \frac{\pi}{2}$.

(h) $\cos x$; $c = \frac{\pi}{4}$.

7. Find the Taylor series up to the term in x^n of the functions at $x = 0$.

(a) $\sin(x^2)$; $n = 10$

(b) $\ln\left(\dfrac{1+x}{1-x}\right)$; $n = 7$

(c) $\dfrac{1}{1+3x^2}$; $n = 6$

(d) $\sqrt{4-x^2}$; $n = 6$

(e) $\dfrac{2-x}{3+x}$; $n = 4$

(f) $\left(\dfrac{1-x^2}{1+x^2}\right)^2$; $n = 6$

(g) $\dfrac{\cos x}{1+x^2}$; $n = 4$

(h) $\ln\left(\dfrac{1+x}{1+x^2}\right)$; $n = 4$

(i) $e^{-2x}\cos(x^2)$; $n = 4$

(j) $\sqrt{\dfrac{1+x^2}{1-x}}$; $n = 3$

8. By considering appropriate Taylor series expansions, evaluate the limits below.

(a) $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+x)}$

(b) $\displaystyle\lim_{x\to 0}\frac{x(1-\cos x)}{1-\sqrt{1-x^3}}$

(c) $\displaystyle\lim_{x\to 0}\frac{x\sin 3x}{x-\ln(1+x)}$

(d) $\displaystyle\lim_{x\to 0}\frac{e^x-x-\cos x}{x\sin x}$

(e) $\displaystyle\lim_{x\to 0}\frac{(1-\cos x)\ln(1+x)}{x-\sin x}$

(f) $\displaystyle\lim_{x\to 0}\frac{\cos 2x-\sqrt{1-4x^2}}{x^4}$

(g) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}+\frac{1}{\ln(1-x)}\right)$

(h) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\tan x}-\frac{1}{e^x-1}\right)$

9. Suppose the Taylor series of $f(x)$ at $x = 0$ is $1 + a_1x + a_2x^2 + a_3x^3 + \cdots$. Find the Taylor series of $\frac{1}{f}$ up to the term in x^3 at $x = 0$ by

(a) expanding the product of the series for f and $\frac{1}{f}$.

(b) finding the first three derivatives of $\frac{1}{f}$.

(c) Find the Taylor series of the following functions up to the term in x^3 at $x = 0$.

(i) $\dfrac{1}{1-x-x^2}$

(ii) $\sec x$

(iii) $\dfrac{1}{1+\sin x}$

(iv) $\dfrac{1}{1+\tan^{-1} x}$

10. Given $y + \dfrac{y^3}{3} = x$.

(a) Find $\dfrac{dy}{dx}$.

(b) Show that $\dfrac{d^2y}{dx^2} = -\dfrac{2y}{(1+y^2)^3}$.

(c) Let $f(x)$ be an infinitely differentiable function which satisfies

$$f(x) + \frac{(f(x))^3}{3} = x, \text{ for all } x \in \mathbb{R}.$$

Find the Taylor series of $f(x)$ about $x = 0$, up to the term in x^3.

Answers:

1. (a) $\frac{1}{2}$ (b) 6 (c) 2 (d) 1 (e) 1 (f) 0 (g) 0 (h) 0 (i) 1 (j) 0 (k) $\frac{1}{2}$ (l) $\frac{1}{2}$ (m) $\frac{1}{2}$ (n) $\frac{\ln 2}{2}$ (o) $\frac{\ln 3}{\ln 2}$

2. (a) 0 (b) 4 (c) $\frac{3}{2}$ (d) 4 (e) $\ln 2$ (f) 1 (g) 1 (h) 3

3. (a) 1 (b) e (c) $\frac{1}{e}$ (d) 1 (e) 1 (f) e (g) $e^{\frac{1}{e}}$ (h) 1 (i) $\frac{e}{2}$ (j) e^6 (k) e^2 (l) 1 (m) e

4. (a) $x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$ (b) $1 + x + \dfrac{x^2}{2}$ (c) $1 + \dfrac{x^2}{2}$ (d) $x + \dfrac{x^2}{3}$

5. (a) $1-2x+3x^2-4x^3+\cdots$
(b) $1-\frac{3x}{2}-\frac{9x^2}{8}-\frac{27x^3}{16}+\cdots$
(c) $x+6x^2+24x^3+\cdots$
(d) $\frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}-\frac{x^3}{81}+\cdots$
(e) $\frac{x}{3}+\frac{3x^2}{4}+\frac{7x^3}{8}+\cdots$
(f) $e^2+e^2x+\frac{e^2x^2}{2}+\frac{e^2x^3x}{6}+\cdots$
(g) $1+\frac{x}{e}-\frac{x^2}{2e^2}+\frac{x^3}{3e^3}+\cdots$
(h) $1-x^2+\cdots$
(i) $x-\frac{13x^3}{6}+\cdots$
(j) $1+2x+\frac{3x^2}{2}+\frac{x^3}{3}+\cdots$
(k) $1+2x+x^2-\frac{x^3}{3}+\cdots$
(l) $1+(\ln 3)x+\frac{(\ln 3)^2x^2}{2}+\frac{(\ln 3)^3x^3}{6}+\cdots$
(m) $x+\frac{x^3}{6}+\cdots$
(n) $x-\frac{x^3}{6}+\cdots$

6. (a) $\frac{1}{2}-\frac{1}{4}(x-1)+\frac{1}{8}(x-1)^2-\frac{1}{16}(x-1)^3+\cdots$
(b) $\frac{1}{4}-\frac{5}{16}(x-1)+\frac{5}{64}(x-1)^2-\frac{5}{256}(x-1)^3+\cdots$
(c) $2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-1)^3+\cdots$
(d) $2+\frac{1}{4}(x-1)-\frac{1}{32}(x-1)^2+\frac{5}{768}(x-1)^3+\cdots$
(e) $e+e(x-1)-\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3+\cdots$
(f) $(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\cdots$
(g) $1-\frac{1}{2}(x-\frac{\pi}{2})^2+\cdots$
(h) $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2+\frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3+\cdots$

7. (a) $x^2-\frac{x^6}{6}+\frac{x^{10}}{120}+\cdots$
(b) $2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\frac{2x^7}{7}+\cdots$
(c) $1-3x^2+9x^4-27x^6+\cdots$
(d) $2-\frac{x^2}{4}-\frac{x^4}{64}-\frac{x^6}{512}+\cdots$
(e) $\frac{1}{2}-\frac{3x}{4}+\frac{3x^2}{8}-\frac{3x^3}{16}+\frac{3x^4}{32}+\cdots$
(f) $1-4x^2+8x^4-12x^6+\cdots$
(g) $1-\frac{3x^2}{2}+\frac{37x^4}{24}+\cdots$
(h) $x-\frac{3x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$
(i) $1-2x+2x^2-\frac{4x^3}{3}+\frac{x^4}{6}+\cdots$
(j) $1+\frac{x}{2}+\frac{7x^2}{8}+\frac{9x^3}{16}+\cdots$

8. (a) 2
(b) 1
(c) 6
(d) 1
(e) 3
(f) $\frac{8}{3}$
(g) 1
(h) $\frac{1}{2}$

9. (a) $1 - a_1 x + (a_1^2 - a_2)x^2 + (2a_1a_2 - a_3 - a_1^3)x^3 + \cdots$

   (b) Same as (a).

   (c) (i) $1 + x + 2x^2 + 3x^3 + \cdots$ (iii) $1 - x + x^2 - \frac{5x^3}{6} + \cdots$

   (ii) $1 + \frac{x^2}{2} + \cdots$ (iv) $1 - x + x^2 - \frac{2x^3}{3} + \cdots$

10. (a) $\frac{1}{1+y^2}$

    (c) $x - \frac{x^3}{3} + \cdots$

Solutions:

1. (a)

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - 1 - \sin x}{x \sin x} &= \lim_{x\to 0} \frac{e^x - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{e^x + \sin x}{\cos x - x \sin x + \cos x} \\ &= \tfrac{1}{2}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x} &= \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x} = \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{\sin x} \\ &= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{\cos x} \\ &= 6\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0} \frac{\ln(1-x^2)}{\ln \cos x} &= \lim_{x\to 0} \frac{-\frac{2x}{1+x^2}}{-\frac{\sin x}{\cos x}} = \lim_{x\to 0} \left(\frac{x}{\sin x}\right)\left(\frac{2\cos x}{1+x^2}\right) \\ &= 2\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^{-1} x}{x} &= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} \\ &= 1\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to 0} \frac{\tan^{-1} x}{1 - \sqrt{1-x}} &= \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x}}} \\ &= 1\end{aligned}$$

(f)

$$\begin{aligned}\lim_{x\to 0} x\ln(1-\cos x) &= \lim_{x\to 0}\frac{\ln(1-\cos x)}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{\sin x}{1-\cos x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0}\frac{-x^2\sin x}{1-\cos x} = \lim_{x\to 0}\frac{-x^2(1+\cos x)}{\sin x} \\ &= 0\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to 0}\sin x\ln x &= \lim_{x\to 0}\frac{\ln x}{\csc x} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\csc x\cot x} \\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x\cos x} \\ &= 0\end{aligned}$$

(h)

$$\begin{aligned}\lim_{x\to 0} x^3(\ln x)^2 &= \lim_{x\to 0}\frac{(\ln x)^2}{\frac{1}{x^3}} = \lim_{x\to 0}\frac{\frac{2\ln x}{x}}{-\frac{3}{x^4}} = \lim_{x\to 0}\frac{-2\ln x}{\frac{3}{x^3}} \\ &= \lim_{x\to 0}\frac{-\frac{2}{x}}{-\frac{9}{x^4}} = \lim_{x\to 0}\frac{2x^3}{9} \\ &= 0\end{aligned}$$

(i)

$$\begin{aligned}\lim_{x\to 0^+} x\ln(1+e^{\frac{1}{x}}) &= \lim_{x\to 0}\frac{\ln(1+e^{\frac{1}{x}})}{\frac{1}{x}} = \lim_{y\to +\infty}\frac{\ln(1+e^y)}{y} = \lim_{y\to +\infty}\frac{e^y}{1+e^y} \\ &= 1\end{aligned}$$

(j)

$$\begin{aligned}\lim_{x\to 0^-} x\ln(1+e^{\frac{1}{x}}) &= \lim_{x\to 0}\frac{\ln(1+e^{\frac{1}{x}})}{\frac{1}{x}} = \lim_{y\to -\infty}\frac{\ln(1+e^y)}{y} \\ &= 0\end{aligned}$$

(k)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) &= \lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)} = \lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x} \\ &= \lim_{x\to 0}\frac{e^x}{e^x+e^x+xe^x} \\ &= \frac{1}{2}\end{aligned}$$

(l)

$$
\begin{aligned}
\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) &= \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}} \\
&= \lim_{x\to 1}\frac{x-1}{x\ln x+x-1} = \lim_{x\to 1}\frac{1}{1+\ln x+1} \\
&= \frac{1}{2}
\end{aligned}
$$

(m)

$$
\begin{aligned}
&\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\sin x}\right) \\
=\;& \lim_{x\to 0}\frac{\sin x-\ln(1+x)}{\sin x\ln(1+x)} = \lim_{x\to 0}\frac{\cos x-\frac{\sin x}{1+x}}{\cos x\ln(1+x)+\frac{\sin x}{1+x}} \\
=\;& \lim_{x\to 0}\frac{(1+x)\cos x-\sin x}{(1+x)\cos x\ln(1+x)+\sin x} \\
=\;& \lim_{x\to 0}\frac{\cos x-(1+x)\sin x-\cos x}{\cos x\ln(1+x)-(1+x)\sin x\ln(1+x)+\cos x+\cos x} \\
=\;& \frac{1}{2}
\end{aligned}
$$

(n)

$$
\begin{aligned}
\lim_{x\to 0}\frac{2^x-1}{\tan 2x} &= \lim_{x\to 0}\frac{2^x\ln 2}{2\sec^2 2x} \\
&= \frac{\ln 2}{2}
\end{aligned}
$$

(o)

$$
\begin{aligned}
\lim_{x\to 0}\frac{3^x-1}{2^x-1} &= \lim_{x\to 0}\frac{3^x\ln 3}{2^x\ln 2} \\
&= \frac{\ln 3}{\ln 2}
\end{aligned}
$$

2. (a)

$$
\begin{aligned}
\lim_{x\to +\infty}\frac{(\ln x)^2}{x} &= \lim_{x\to +\infty}\frac{2\ln x}{x} = \lim_{x\to +\infty}\frac{2}{x} \\
&= 0
\end{aligned}
$$

(b)

$$
\begin{aligned}
\lim_{x\to +\infty}\frac{\ln(1+x^4)}{\ln(1+x)} &= \lim_{x\to +\infty}\frac{\frac{4x^3}{1+x^4}}{\frac{1}{1+x}} = \lim_{x\to +\infty}\frac{4x^3(1+x)}{1+x^4} \\
&= 4
\end{aligned}
$$

(c)

$$\lim_{x\to+\infty}\frac{\ln(2x^3-5x^2+3)}{\ln(4x^2+x-7)} = \lim_{x\to+\infty}\frac{\frac{6x^2-10x}{2x^3-5x^2+3}}{\frac{8x+1}{4x^2+x-7}} = \lim_{x\to+\infty}\frac{(6x^2-10x)(4x^2+x-7)}{(8x+1)(2x^3-5x^2+3)} = \frac{3}{2}$$

(d)

$$\lim_{x\to+\infty}\frac{\ln(e^{4x}+x^3)}{\ln(e^x+x)} = \lim_{x\to+\infty}\frac{\frac{4e^{4x}+3x^2}{e^4x+x^3}}{\frac{e^x+1}{e^x+x}} = \lim_{x\to+\infty}\frac{(e^x+x)(4e^{4x}+3x^2)}{(e^x+1)(e^4x+x^3)} = \lim_{x\to+\infty}\frac{(1+xe^{-x})(4+3x^2e^{-4x})}{(1+e^{-x})(1+x^3)e^{-4x}} = 4$$

(e)

$$\lim_{x\to+\infty} x(2^{\frac{1}{x}}-1) = \lim_{x\to+\infty}\frac{2^{\frac{1}{x}}-1}{\frac{1}{x}} = \lim_{y\to0^+}\frac{2^y-1}{y} = \lim_{y\to0^+}\frac{2^y\ln 2}{1} = \ln 2$$

(f)

$$\lim_{x\to+\infty} x\sin^{-1}\left(\frac{1}{x}\right) = \lim_{x\to+\infty}\frac{\sin^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{y\to0^+}\frac{\sin^{-1}y}{y} = \lim_{y\to0^+}\frac{1}{\sqrt{1-y^2}} = 1$$

(g)

$$\lim_{x\to+\infty} x\left(\frac{\pi}{2}-\tan^{-1}x\right) = \lim_{x\to+\infty}\frac{\frac{\pi}{2}-\tan^{-1}x}{\frac{1}{x}} = \lim_{x\to+\infty}\frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{x^2}{1+x^2} = 1$$

(h)

$$\begin{aligned}\lim_{x\to+\infty} x\ln\left(1+\frac{3}{x}\right) &= \lim_{x\to+\infty}\frac{\ln(1+\frac{3}{x})}{\frac{1}{x}} = \lim_{y\to 0^+}\frac{\ln(1+3y)}{y}\\ &= \lim_{y\to 0^+}\frac{3}{1+3y}\\ &= 3\end{aligned}$$

3. (a)

$$\begin{aligned}\ln\left(\lim_{x\to 0} x^{x^2}\right) &= \lim_{x\to 0}\ln(x^{x^2}) = \lim_{x\to 0} x^2\ln x\\ &= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0}\frac{-x^2}{2}\\ &= 0\end{aligned}.$$

Thus $\lim_{x\to 0} x^{x^2} = e^0 = 1$.

(b)

$$\begin{aligned}\ln\left(\lim_{x\to 0} x^{\frac{1}{1+\ln x}}\right) &= \lim_{x\to 0}\frac{\ln x}{1+\ln x} = \lim_{x\to 0}\frac{1}{\frac{1}{\ln x}+1}\\ &= 1\end{aligned}.$$

Thus $\lim_{x\to 0} x^{\frac{1}{1+\ln x}} = e^1 = e$.

(c)

$$\begin{aligned}\ln\left(\lim_{x\to 1} x^{\frac{1}{1-x}}\right) &= \lim_{x\to 1}\frac{\ln x}{1-x} = \lim_{x\to 1}\frac{\frac{1}{x}}{-1}\\ &= -1\end{aligned}.$$

Thus $\lim_{x\to 0} x^{x^2} = e^{-1}$.

(d)

$$\begin{aligned}\ln\left(\lim_{x\to 0}(\sin x)^x\right) &= \lim_{x\to 0} x\ln\sin x = \lim_{x\to 0}\frac{\ln\sin x}{\frac{1}{x}}\\ &= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}\\ &= 0\end{aligned}.$$

Thus $\lim_{x\to 0}(\sin x)^x = e^0 = 1$.

(e)

$$\begin{aligned}\ln\left(\lim_{x\to 0}(\sin x)^{\sin x}\right) &= \lim_{x\to 0}\sin x\ln\sin x = \lim_{x\to 0}\frac{\ln\sin x}{\csc x}\\ &= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{-\csc x\cot x} = \lim_{x\to 0}(-\sin^3 x)\\ &= 0\end{aligned}.$$

Thus $\lim_{x\to 0}(\sin x)^{\sin x} = e^0 = 1$.

(f)

$$\begin{aligned} \ln\left(\lim_{x\to 0}(1+\sin x)^{\frac{1}{\tan x}}\right) &= \lim_{x\to 0}\frac{\ln(1+\sin x)}{\tan x} = \lim_{x\to 0}\frac{\frac{\cos x}{1+\sin x}}{\sec^2 x} \\ &= \lim_{x\to 0}\frac{\cos^3 x}{1+\sin x} = 1 \end{aligned}.$$

Thus $\lim_{x\to 0}(1+\sin x)^{\frac{1}{\tan x}} = e^1 = e$.

(g)

$$\begin{aligned} \ln\left(\lim_{x\to 0}\frac{(e+x)^{\frac{1}{x}}}{e^{\frac{1}{x}}}\right) &= \lim_{x\to 0}\ln\left(1+\frac{x}{e}\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{\ln(1+\frac{x}{e})}{x} \\ &= \frac{1}{e} \end{aligned}. \text{ Thus}$$

$\lim_{x\to 0}\frac{(e+x)^{\frac{1}{x}}}{e^{\frac{1}{x}}} = e^{\frac{1}{e}}$.

(h)

$$\begin{aligned} \lim_{x\to 0}\frac{(1+x)^x-1}{x^2} &= \lim_{x\to 0}\frac{e^{x\ln(1+x)}-1}{x^2} = \lim_{x\to 0}\frac{e^{x\ln(1+x)}(\frac{x}{1+x}+\ln(1+x))}{2x} \\ &= \lim_{x\to 0}e^{x\ln(1+x)}\lim_{x\to 0}\frac{(x+(1+x)\ln(1+x))}{2x(1+x)} \\ &= \lim_{x\to 0}\frac{(1+(1+\ln(1+x)))}{2+4x} \\ &= 1 \end{aligned}.$$

(i)

$$\begin{aligned} \lim_{x\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x} &= \lim_{x\to 0}\frac{e-e^{\frac{\ln(1+x)}{x}}}{x} = \lim_{x\to 0}(-e^{\frac{\ln(1+x)}{x}})\frac{\frac{x}{1+x}-\ln(1+x)}{x^2} \\ &= e\lim_{x\to 0}\frac{(1+x)\ln(1+x)-x}{x^2(1+x)} = e\lim_{x\to 0}\frac{\ln(1+x)}{2x+3x^2} \\ &= e\lim_{x\to 0}\frac{\frac{1}{1+x}}{2+6x} \\ &= \frac{e}{2} \end{aligned}.$$

(j)

$$\begin{aligned} \ln\left(\lim_{x\to+\infty}(1+5e^{2x})^{\frac{3}{x}}\right) &= \lim_{x\to+\infty}\frac{3\ln(1+5e^{2x})}{x} = \lim_{x\to+\infty}\frac{30e^{2x}}{1+5e^{2x}} \\ &= 6 \end{aligned}.$$

Thus $\lim_{x\to+\infty}(1+5e^{2x})^{\frac{3}{x}} = e^6$.

(k)

$$\ln\left(\lim_{x\to+\infty}(1+3x)^{\frac{2}{\ln x}}\right) = \lim_{x\to+\infty}\frac{2\ln(1+3x)}{\ln x} = \lim_{x\to+\infty}\frac{\frac{6}{1+3x}}{\frac{1}{x}} .$$
$$= \lim_{x\to+\infty}\frac{6x}{1+3x} = 2$$

Thus $\lim_{x\to+\infty}(1+3x)^{\frac{1}{\ln x}} = e^2$.

(l) Since $|\sin(e^x)| \le 1$ and $|e^{\sin x}| \le e$ are bounded, we have $\lim_{x\to+\infty}\frac{\sin(e^x)}{x} = \lim_{x\to+\infty}\frac{e^{\sin x}}{x} = 0$. Therefore $\lim_{x\to+\infty}\frac{x+\sin(e^x)}{x+e^{\sin x}} = \lim_{x\to+\infty}\frac{1+\frac{\sin(e^x)}{x}}{1+\frac{e^{\sin x}}{x}} = 1$

(m) Since $|\cos x| \le 1$ is bounded, we have $\lim_{x\to+\infty}\frac{\cos x}{e^x} = 0$. Thus

$\lim_{x\to+\infty}(e^x+\cos x)^{\frac{1}{x}} = \lim_{x\to+\infty} e\left(1+\frac{\cos x}{e^x}\right)^{\frac{1}{x}} = e.$

On the other hand,

$\lim_{x\to+\infty}(x^2+1)^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\frac{\ln(1+x^2)}{x}} = \lim_{x\to+\infty} e^{\frac{2x}{1+x^2}} = e^0 = 1.$

Therefore $\lim_{x\to+\infty}\left(\frac{e^x+\cos x}{x^2+1}\right)^{\frac{1}{x}} = e.$

4. (a)

k	0	1	2	3
$f^{(k)}(x)$	$(1+x)\ln(1+x)$	$1+\ln(1+x)$	$\frac{1}{1+x}$	$-\frac{1}{(1+x)^2}$
$f^{(k)}(0)$	0	1	1	-1

The Taylor polynomial of $(1+x)\ln(1+x)$ of degree 3 at $x=0$ is

$$0+(1)x+\frac{(1)x^2}{2!}+\frac{(-1)x^3}{3!} = x+\frac{x^2}{2}-\frac{x^3}{6}$$

(b)

k	0	1	2	3
$f^{(k)}(x)$	$e^{\sin x}$	$e^{\sin x}\cos x$	$e^{\sin x}(\cos^2 x-\sin x)$	$e^{\sin x}\cos x(\cos^2 x-3\sin x-1)$
$f^{(k)}(0)$	1	1	1	0

The Taylor polynomial of $e^{\sin x}$ of degree 3 at $x=0$ is

$$1+(1)x+\frac{(1)x^2}{2!}+\frac{(0)x^3}{3!} = 1+x+\frac{x^2}{2}$$

(c)

k	0	1	2	3
$f^{(k)}(x)$	$\sec x$	$\sec x \tan x$	$2\sec^3 x - \sec x$	$6\sec^3 x \tan x - \sec x \tan x$
$f^{(k)}(0)$	1	0	1	0

The Taylor polynomial of $\sec x$ of degree 3 at $x = 0$ is

$$1 + (0)x + \frac{(1)x^2}{2!} + \frac{(0)x^3}{3!} = 1 + \frac{x^2}{2}$$

(d)

k	0	1	2	3
$f^{(k)}(x)$	$\tan x$	$\sec^2 x$	$2\sec^2 x \tan x$	$2\sec^4 x + 4\sec^2 x \tan^2 x$
$f^{(k)}(0)$	0	1	0	2

The Taylor polynomial of $\tan x$ of degree 3 at $x = 0$ is

$$0 + (1)x + \frac{(0)x^2}{2!} + \frac{(2)x^3}{3!} = x + \frac{x^2}{3}$$

5. (a) The Taylor series of $\frac{1}{1+x)^2} = (1+x)^{-2}$ at $x = 0$ is

$$\begin{aligned} & 1 - 2x + \frac{(-2)(-3)x^2}{2!} + \frac{(-2)(-3)(-4)x^3}{3!} + \cdots \\ = \; & 1 - 2x + 3x^2 - 4x^3 + \cdots \end{aligned}$$

(b) The Taylor series of $\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$ at $x = 0$ is

$$\begin{aligned} & 1 - \frac{1}{2}(3x) + \frac{(\frac{1}{2})(-\frac{1}{2})(3x)^2}{2!} - \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(3x)^3}{3!} + \cdots \\ = \; & 1 - \frac{3x}{2} - \frac{9x^2}{8} - \frac{27x^3}{16} + \cdots \end{aligned}$$

(c) The Taylor series of $\frac{x}{(1-2x)^3} = x(1-2x)^{-3}$ at $x = 0$ is

$$\begin{aligned} & x(1 - (-3)(2x) + \frac{(-3)(-4)(2x)^2}{2!} + \cdots) \\ = \; & x + 6x^2 + 24x^3 + \cdots \end{aligned}$$

(d) The Taylor series of $\frac{1}{x+3} = \frac{1}{3}(1 + \frac{x}{3})^{-1}$ at $x = 0$ is

$$\begin{aligned} & \frac{1}{3}\left(1 - \frac{x}{3} + \left(\frac{x}{3}\right)^2 - \left(\frac{x}{3}\right)^3 + \cdots\right) \\ = \; & \frac{1}{3} - \frac{x}{9} + \frac{x^2}{27} - \frac{x^3}{81} + \cdots \end{aligned}$$

(e) The Taylor series of $\frac{x}{(x-1)(x-2)} = \frac{1}{1-x} - \frac{2}{2-x} = \frac{1}{1-x} - \frac{1}{1-\frac{x}{2}}$ at $x = 0$ is

$$\begin{aligned} & (1 + x + x^2 + x^3 + \cdots) - \left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\right) \\ = \ & \frac{x}{2} + \frac{3x^2}{4} + \frac{7x^3}{8} + \cdots \end{aligned}$$

(f) The Taylor series of $e^{x+2} = e^2 e^x$ at $x = 0$ is

$$\begin{aligned} & e^2(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ = \ & e^2 + e^2 x + \frac{e^2 x^2}{2} + \frac{e^2 x^3}{6} + \cdots \end{aligned}$$

(g) The Taylor series of $\ln(e + x) = 1 + \ln(1 + \frac{x}{e})$ at $x = 0$ is

$$\begin{aligned} & 1 + \left(\frac{x}{e} - \frac{(\frac{x}{e})^2}{2} + \frac{(\frac{x}{e})^3}{3} + \cdots\right) \\ = \ & 1 + \frac{x}{e} - \frac{x^2}{2e^2} + \frac{x^3}{3e^3} + \cdots \end{aligned}$$

(h) The Taylor series of $\cos^2 x = \frac{1+\cos 2x}{2}$ at $x = 0$ is

$$\begin{aligned} & \frac{1}{2} + \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \cdots\right) \\ = \ & 1 - x^2 + \cdots \end{aligned}$$

(i) The Taylor series of $\sin x \cos 2x = \frac{1}{2}(\sin 3x - \sin x)$ at $x = 0$ is

$$\begin{aligned} & \frac{1}{2}\left(\left(3x - \frac{(3x)^3}{3!} + \cdots\right) - \left(x - \frac{x^3}{3!} + \cdots\right)\right) \\ = \ & x - \frac{13x^3}{6} + \cdots \end{aligned}$$

(j) The Taylor series of $e^{2x} \cos x$ at $x = 0$ is

$$\begin{aligned} & \left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \cdots\right) \\ = \ & \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots\right)\left(1 - \frac{x^2}{2} + \cdots\right) \\ = \ & 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{3} + \cdots \end{aligned}$$

(k) The Taylor series of $(1+\sin x)^2 = 1+2\sin x+\sin^2 x = \frac{3}{2}+2\sin x-\frac{\cos 2x}{2}$ at $x=0$ is

$$\begin{aligned} & \frac{3}{2}+2\left(x-\frac{x^3}{6}+\cdots\right)-\frac{1}{2}\left(1-2x^2+\cdots\right) \\ = \;& 1+2x+x^2-\frac{x^3}{3}+\cdots \end{aligned}$$

(l) The Taylor series of $3^x = e^{(\ln 3)x}$ at $x=0$ is

$$1+(\ln 3)x+\frac{(\ln 3)^2x^2}{2}+\frac{(\ln 3)^3x^3}{6}+\cdots$$

(m) Let $f(x)=\sin^{-1} x$. The Taylor series of $f'(x)=(1-x^2)^{-\frac{1}{2}}$ at $x=0$ is

$$1+\frac{x^2}{2}+\cdots$$

Now $f(0)=0$. The Taylor series of $f(x)$ at $x=0$ is

$$x+\frac{x^3}{6}+\cdots$$

(n) Let $f(x)=\ln(x+\sqrt{1+x^2})$. The Taylor series of $f'(x)=(1+x^2)^{-\frac{1}{2}}$ at $x=0$ is

$$1-\frac{x^2}{2}+\cdots$$

Now $f(0)=0$. The Taylor series of $f(x)$ at $x=0$ is

$$x-\frac{x^3}{6}+\cdots$$

6. (a) The Taylor series of $\frac{1}{1+x}=\frac{1}{2+(x-1)}=\frac{1}{2}\left(\frac{1}{1+\frac{x-1}{2}}\right)$ at $x=1$ is

$$\begin{aligned} & \frac{1}{2}\left(1-\frac{x-1}{2}+\left(\frac{x-1}{2}\right)^2-\left(\frac{x-1}{2}\right)^3+\cdots\right) \\ = \;& \frac{1}{2}-\frac{x-1}{4}+\frac{(x-1)^2}{8}-\frac{(x-1)^3}{16}+\cdots \end{aligned}$$

(b) The Taylor series of

$$\frac{2-x}{3+x} = -1 + \frac{5}{3+x} = -1 + \frac{5}{4+(x-1)} = -1 + \frac{5}{4}\left((\frac{1}{1+\frac{x-1}{4}} \right)$$

at $x = 1$ is

$$\begin{aligned} & -1 + \frac{5}{4}\left(1 - \frac{x-1}{4} + \left(\frac{x-1}{4}\right)^2 - \left(\frac{x-1}{4}\right)^3 + \cdots\right) \\ = \ & \frac{1}{4} - \frac{5(x-1)}{16} + \frac{5(x-1)^2}{64} - \frac{5(x-1)^3}{256} + \cdots \end{aligned}$$

(c) The Taylor series of $\sqrt{x} = \sqrt{4+(x-4)} = 2(1+\frac{x-4}{4})^{\frac{1}{2}}$ at $x = 4$ is

$$2\left(1 + \tfrac{1}{2}\left(\tfrac{x-4}{4}\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\tfrac{x-4}{4}\right)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\left(\tfrac{x-4}{4}\right)^3 + \cdots\right)$$
$$= 2 + \tfrac{x-4}{4} - \tfrac{(x-4)^2}{64} + \tfrac{(x-4)^2}{512} + \cdots$$

(d) The Taylor series of $\sqrt[3]{5+3x} = (8+3(x-1))^{\frac{1}{3}} = 2(1+\frac{3(x-1)}{8})^{\frac{1}{3}}$ at $x = 1$ is

$$2\left(1 + \frac{1}{3}\cdot\frac{3(x-1)}{8} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\cdot\frac{9(x-1)^2}{64} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\cdot\frac{27(x-1)^3}{512} + \cdots\right)$$
$$= 2 + \frac{x-1}{4} - \frac{(x-1)^2}{32} + \frac{5(x-1)^3}{768} + \cdots$$

(e) The Taylor series of $e^x = e \cdot e^{x-1}$ at $x = 1$ is

$$\begin{aligned} & e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right) \\ = \ & e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{6} + \cdots \end{aligned}$$

(f) The Taylor series of $\ln x = \ln(1+(x-1))$ at $x = 1$ is

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$$

(g) The Taylor series of $\sin^x = \cos(x - \frac{\pi}{2})$ at $x = \frac{\pi}{2}$ is

$$1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \cdots$$

(h) The Taylor series of

$$\begin{aligned}\cos x &= \cos((x - \frac{\pi}{4}) + \frac{\pi}{4}) = \cos\frac{\pi}{4}\cos(x - \frac{\pi}{4}) - \sin\frac{\pi}{4}\sin(x - \frac{\pi}{4}) \\ &= \frac{\sqrt{2}}{2}\cos(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{2}\sin(x - \frac{\pi}{4})\end{aligned}$$

at $x = \frac{\pi}{4}$ is

$$\begin{aligned}&\frac{\sqrt{2}}{2}\left(1 - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \cdots\right) - \frac{\sqrt{2}}{2}\left(\left(x - \frac{\pi}{4}\right) - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^4 + \cdots\right) \\ = \ &\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + \cdots\end{aligned}$$

7. (a) The Taylor series of $\sin(x^2)$ at $x = 0$ is
$x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \cdots = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} + \cdots$

(b) The Taylor series of $\ln(\frac{1+x}{1-x}) = \ln(1+x) - \ln(1-x)$ at $x = 0$ is

$$\begin{aligned}&(x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \cdots) - (-x - \tfrac{x^2}{2} - \tfrac{x^3}{3} - \tfrac{x^4}{4} + \cdots) \\ = \ &2x + \tfrac{2x^3}{3} + \tfrac{2x^5}{5} + \tfrac{2x^7}{7} + \cdots\end{aligned}.$$

(c) The Taylor series of $\frac{1}{1+3x^2}$ at $x = 0$ is

$$\begin{aligned}&1 - 3x^2 + (3x^2)^2 - (3x^2)^3 + \cdots \\ = \ &1 - 3x^2 + 9x^4 - 27x^6 + \cdots\end{aligned}.$$

(d) The Taylor series of $\sqrt{4 - x^2} = 2(1 - (\frac{x}{2})^2)^{\frac{1}{2}}$ at $x = 0$ is

$$\begin{aligned}&2(1 - \tfrac{1}{2}(\tfrac{x}{2})^2 + \tfrac{\frac{1}{2}(-\frac{1}{2})}{2!}(\tfrac{x}{2})^4 - \tfrac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(\tfrac{x}{2})^6 + \cdots \\ = \ &2 - \tfrac{x^2}{4} - \tfrac{x^4}{64} - \tfrac{x^6}{512} + \cdots\end{aligned}.$$

(e) The Taylor series of $\frac{1-x}{2+x} = -1 + \frac{3}{2+x} = -1 + \frac{3}{2}(\frac{1}{1+\frac{x}{2}})$ at $x = 0$ is

$$\begin{aligned}&-1 + \tfrac{3}{2}(1 - \tfrac{x}{2} + \tfrac{x^2}{4} - \tfrac{x^3}{8} + \tfrac{x^4}{16} + \cdots \\ = \ &\tfrac{1}{2} - \tfrac{3x}{4} + \tfrac{3x^2}{8} - \tfrac{3x^3}{16} + \tfrac{3x^4}{32} + \cdots\end{aligned}.$$

(f) The Taylor series of $(\frac{1-x^2}{1+x^2})^2 = (1-x^2)^2(1+x^2)^{-2}$ at $x = 0$ is

$$\begin{aligned}&(1 - 2x^2 + x^4)(1 + 2x^2 + \tfrac{(-2)(-3)x^4}{2!} + \tfrac{(-2)(-3)(-4)x^6}{3!} + \cdots) \\ = \ &(1 - 2x^2 + x^4)(1 - 2x^2 + 3x^4 - 4x^6 + \cdots) \\ = \ &1 - 4x^2 + 8x^4 - 12x^6 + \cdots\end{aligned}.$$

(g) The Taylor series of $\frac{\cos x}{1+x^2}$ at $x=0$ is

$$\begin{aligned} & (1-\tfrac{x^2}{2}+\tfrac{x^4}{24}+\cdots)(1-x^2+x^4+\cdots) \\ = & 1-\tfrac{3x^2}{2}+\tfrac{37x^4}{24}+\cdots \end{aligned}.$$

(h) The Taylor series of $\ln(\frac{1+x}{1+x^2})=\ln(1+x)-\ln(1+x^2)$ at $x=0$ is

$$\begin{aligned} & (x-\tfrac{x^2}{2}+\tfrac{x^3}{3}-\tfrac{x^4}{4}+\cdots)-(x^2-\tfrac{x^4}{2}+\cdots) \\ = & x-\tfrac{3x^2}{2}+\tfrac{x^3}{3}+\tfrac{x^4}{4}+\cdots \end{aligned}.$$

(i) The Taylor series of $e^{-2x}\cos(x^2)$ at $x=0$ is

$$\begin{aligned} & (1-2x+\tfrac{4x^2}{2!}-\tfrac{8x^3}{3!}+\tfrac{16x^4}{4!}+\cdots)(1-\tfrac{x^4}{2}+\cdots) \\ = & 1-2x+2x^2-\tfrac{4x^3}{3}+\tfrac{x^4}{6}+\cdots \end{aligned}.$$

(j) The Taylor series of $\sqrt{\frac{1+x^2}{1-x}}=(1+x^2)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ at $x=0$ is

$$\begin{aligned} & (1+\tfrac{x^2}{2}+\cdots)(1-(-\tfrac{1}{2})x+\tfrac{(-\frac{1}{2})(-\frac{3}{2})x^2}{2!}-\tfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})x^3}{3!}+\cdots) \\ = & (1+\tfrac{x^2}{2}+\cdots)(1+\tfrac{x}{2}+\tfrac{3x^2}{8}+\tfrac{5x^3}{16}+\cdots) \\ = & 1+\tfrac{x}{2}+\tfrac{7x^2}{8}+\tfrac{9x^3}{16}+\cdots \end{aligned}.$$

8. (a)

$$\begin{aligned} \lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+x)} &= \lim_{x\to 0}\frac{(1+2x+\cdots)-1}{x+\cdots}=\lim_{x\to 0}\frac{2x+\cdots}{x+\cdots} \\ &= 2 \end{aligned}.$$

(b)

$$\begin{aligned} \lim_{x\to 0}\frac{x(1-\cos x)}{1-\sqrt{1-x^2}} &= \lim_{x\to 0}\frac{x(1-(1-\frac{x^2}{2}+\cdots))}{1-(1-\frac{x^3}{2}+\cdots)}=\lim_{x\to 0}\frac{\frac{x^3}{2}+\cdots}{\frac{x^3}{2}+\cdots} \\ &= 1 \end{aligned}.$$

(c)

$$\begin{aligned} \lim_{x\to 0}\frac{x\sin 3x}{x-\ln(1+x)} &= \lim_{x\to 0}\frac{x(3x+\cdots)}{x-(x-\frac{x^2}{2}+\cdots)}=\lim_{x\to 0}\frac{3x^2+\cdots}{\frac{x^2}{2}+\cdots} \\ &= 6 \end{aligned}.$$

(d)

$$\begin{aligned} \lim_{x\to 0}\frac{e^x-x-\cos x}{x\sin x} &= \lim_{x\to 0}\frac{(1+x+\frac{x^2}{2}+\cdots)-x-(1-\frac{x^2}{2}+\cdots)}{x(x+\cdots)} \\ &= \lim_{x\to 0}\frac{x^2+\cdots}{x^2+\cdots} \\ &= 1 \end{aligned}.$$

(e)

$$
\begin{aligned}
\lim_{x\to 0}\frac{(1-\cos x)\ln(1+x)}{x-\sin x} &= \lim_{x\to 0}\frac{(1-(1-\frac{x^2}{2}+\cdots))(x+\cdots)}{x-(x-\frac{x^3}{6}+\cdots)} \\
&= \lim_{x\to 0}\frac{\frac{x^3}{2}+\cdots}{\frac{x^3}{6}+\cdots} \\
&= 3
\end{aligned}
.
$$

(f)

$$
\begin{aligned}
&\lim_{x\to 0}\frac{\cos 2x-\sqrt{1-4x^2}}{x^4} \\
=\;& \lim_{x\to 0}\frac{(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\cdots)-(1-\frac{1}{2}(4x^2)+(\frac{1}{2})(-\frac{1}{2})\frac{(4x^2)^2}{2!}+\cdots)}{x^4} \\
=\;& \lim_{x\to 0}\frac{(1-2x^2+\frac{2x^4}{3}+\cdots)-(1-2x^2-2x^4+\cdots)}{x^4} \\
=\;& \lim_{x\to 0}\frac{\frac{8x^4}{3}+\cdots}{x^4} \\
=\;& \tfrac{8}{3}
\end{aligned}
.
$$

(g)

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}+\frac{1}{\ln(1-x)}\right) &= \lim_{x\to 0}\frac{\ln(1+x)+\ln(1-x)}{\ln(1+x)\ln(1-x)} \\
&= \lim_{x\to 0}\frac{(x-\frac{x^2}{2}+\cdots)+(-x-\frac{x^2}{2}+\cdots)}{(x+\cdots)(-x+\cdots)} \\
&= \lim_{x\to 0}\frac{-x^2+\cdots}{-x^2+\cdots} \\
&= 1
\end{aligned}
.
$$

(h)

$$
\begin{aligned}
&\lim_{x\to 0}\left(\frac{1}{\tan x}-\frac{1}{e^x-1}\right) \\
=\;& \lim_{x\to 0}\frac{\cos x(e^x-1)-\sin x}{\sin x(e^x-1)} \\
=\;& \lim_{x\to 0}\frac{(1-\frac{x^2}{2}+\cdots)(x+\frac{x^2}{2}+\cdots)-(x-\frac{x^3}{6}+\cdots)}{(x-\frac{x^3}{6}+\cdots)(x+\cdots)} \\
=\;& \lim_{x\to 0}\frac{\frac{x^2}{2}+\cdots}{x^2+\cdots} \\
=\;& \tfrac{1}{2}
\end{aligned}
.
$$

9. (a) Suppose the Taylor series of $\frac{1}{f}$ is $b_0+b_1x+b_2x^2+b_3x^3+\cdots$. Then

the Taylor series of $f \cdot \frac{1}{f} = 1$ is

$$\begin{aligned} & (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) \\ = \ & b_0 + (b_1 + a_1 b_0)x + (b_2 + a_1 b_1 + a_2 b_0)x^2 + (b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3 + \cdots \end{aligned}$$

It follows that

$$\begin{cases} b_0 = 1 \\ b_1 + a_1 b_0 = 0 \\ b_2 + a_1 b_1 + a_2 b_0 = 0 \\ b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 0 \end{cases}$$

which implies

$$\begin{cases} b_0 = 1 \\ b_1 = -a_1 b0 = -a_1 \\ b_2 = -a_1 b_1 - a_2 b_0 = -a_1(-a_1) - a_2 = a_1^2 - a_2 \\ b_3 = -a_1 b_2 - a_2 b_1 - a_3 b_0 = -a_1(a_1^2 - a_2) - a_2(-a_1) - a_3 = -a_1^3 + 2a_1 a_2 - a_3 \end{cases}$$

Thus the Taylor series of $\frac{1}{f}$ at $x = 0$ is

$$1 - a_1 x + (a_1^2 - a_2)x^2 + (-a_1^3 + 2a_1 a_2 - a_3)x^3 + \cdots$$

(b) Let $g(x) = \frac{1}{f(x)}$ Then derivatives of $g(x)$ are

$$\begin{cases} g(x) = \dfrac{1}{f} \\ g'(x) = -\dfrac{f'}{f^2} \\ g''(x) = -\dfrac{f^2 f'' - f'(2ff')}{f^4} = \dfrac{2f'^2 - ff''}{f^3} \\ g^{(3)} = \dfrac{f^3(4f'f'' - f'f'' - ff^{(3)}) - (2f'^2 - ff'')(3f^2 f')}{f^6} = \dfrac{6ff'f'' - f^2 f^{(3)} - 6f'^3}{f^4} \end{cases}$$

Since the Taylor series of $f(x)$ is $1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, we see that $f(0) = 1$, $f'(0) = a_1$, $f''(0) = 2a_2$ and $f^{(3)}(0) = 6a_3$. Thus

$$\begin{cases} g(0) = 1 \\ g'(0) = -a_1 \\ g''(0) = 2a_1^2 - 2a_2 \\ \ g^{(3)} = 12a_1 a_2 - 6a_3 - 6a_1^3 \end{cases}$$

Therefore the Taylor series of $g(x) = \frac{1}{f(x)}$ at $x = 0$ is

$$\begin{aligned} & g(0) + g'(0)x + \frac{g''(0)x^2}{2!} + \frac{g^{(3)}(0)x^2}{3!} + \cdots \\ = \;& 1 - a_1 x + (a_1^2 - a_2)x^2 + (2a_1a_2 - a_3 - a_1^3)x^3 + \cdots \end{aligned}$$

(i) The Taylor serious of $1 - x - x^2$ at $x = 0$ is $1 - x - x^2$. Thus the Taylor serious of $\frac{1}{1-x-x^2}$ at $x = 0$ is

$$\begin{aligned} & 1 - a_1 x + (a_1^2 - a_2)x^2 + (2a_1a_2 - a_3 - a_1^3)x^3 + \cdots \\ = \;& 1 - (-1)x + ((-1)^2 - (-1))x^2 + (2(-1)(-1) - 0 - (-1)^3)x^3 + \cdots \\ = \;& 1 + x + 2x^2 + 3x^3 + \cdots \end{aligned}$$

(ii) The Taylor serious of $\cos x$ at $x = 0$ is $1 - \frac{x^2}{2} + \cdots$. Thus the Taylor serious of $\sec x = \frac{1}{\cos x}$ at $x = 0$ is

$$\begin{aligned} & 1 - a_1 x + (a_1^2 - a_2)x^2 + (2a_1a_2 - a_3 - a_1^3)x^3 + \cdots \\ = \;& 1 - (0)x + (0^2 - (-\frac{1}{2}))x^2 + (2(0)(-\frac{1}{2}) - 0 - 0^3)x^3 + \cdots \\ = \;& 1 + \frac{x}{2} + \cdots \end{aligned}$$

i. The Taylor serious of $1 + \sin x$ at $x = 0$ is $1 + x - \frac{x^3}{6} + \cdots$. Thus the Taylor series of $\frac{1}{1+\sin x}$ at $x = 0$ is

$$\begin{aligned} & 1 - a_1 x + (a_1^2 - a_2)x^2 + (2a_1a_2 - a_3 - a_1^3)x^3 + \cdots \\ = \;& 1 - (1)x + (1^2 - 0)x^2 + (2(1)(0) - (-\frac{1}{6}) - 1^3)x^3 + \cdots \\ = \;& 1 - x + x^2 - \frac{5x^3}{6} + \cdots \end{aligned}$$

ii. The Taylor serious of $1 + \tan^{-1} x$ at $x = 0$ is $1 + x - \frac{x^3}{3} + \cdots$. Thus the Taylor series of $\frac{1}{1+\tan^{-1} x}$ at $x = 0$ is

$$\begin{aligned} & 1 - a_1 x + (a_1^2 - a_2)x^2 + (2a_1a_2 - a_3 - a_1^3)x^3 + \cdots \\ = \;& 1 - (1)x + (1^2 - 0)x^2 + (2(1)(0) - (-\frac{1}{3}) - 1^3)x^3 + \cdots \\ = \;& 1 - x + x^2 - \frac{2x^3}{3} + \cdots \end{aligned}$$

10. (a)

$$\begin{aligned}\frac{dy}{dx}+y^2\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= \frac{1}{1+y^2}\end{aligned}$$

(b)

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{2y}{(1+y^2)^2}\cdot\frac{dy}{dx}\\ &= -\frac{2y}{(1+y^2)^2}\cdot\frac{1}{1+y^2}\\ &= -\frac{2y}{(1+y^2)^3}\end{aligned}$$

(c) Let $y=f(x)$. Putting $x=0$, we have $f(0)+\frac{(f(0))^3}{3}=0$ which implies $f(0)=0$. By (a) and (b), we have

$$\begin{aligned}f'(0) &= \frac{1}{1+(f(0))^2}=1\\ f''(0) &= -\frac{2f(0)}{1+(f(0))^3}=0\end{aligned}$$

Moreover

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{2y(3(1+y^2)^2)(2y)-2(1+y^2)^3}{(1+y^2)^6}\cdot\frac{dy}{dx}\\ &= \frac{12y^2-2(1+y^2)}{(1+y^2)^4}\cdot\frac{1}{1+y^2}\\ &= \frac{10y^2-2}{(1+y^2)^5}\end{aligned}$$

Thus $f^{(3)}(0)=-2$. Therefore the Taylor serious of $f(x)$ at $x=0$ is

$$f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f^{(3)}(0)x^3}{3!}+\cdots=x-\frac{x^3}{3}+\cdots.$$