MATH4250 Game Theory, 2016-2017 Term 2
Mid-term Examination
Time allowed: 90 mins

Answer all questions.

1. (6 marks) Let $\oplus$ denotes the nim-sum.
   (a) Find x if $x \oplus 13 \oplus 23 = 28$.
   (b) Find all winning moves of the game of nim from the position $(13, 23, 28)$.

2. (10 marks) There are two piles of chips on the table. Two players remove the chips from the table alternatively. In each turn, a player may either remove 1 or 2 chips from one of the piles, or move 1 or 2 chips from the second pile to the first pile. The player who removes the last chip wins. Let $g(x, y)$ be the Sprague-Grundy function of the game, where x and y are the number of chips in the first and second pile respectively.
   (a) Find $g(5, 1)$, $g(2, 3)$ and $g(99, 100)$.
   (b) Find all winning moves from the position $(8, 5)$.
   (c) Write down a guess of the set of P-positions.
   (d) Prove your assertion in (c).

3. (10 marks) Consider the following 3 games.

Game 1:	1-pile nim
Game 2:	Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6\}$
Game 3:	Subtraction game with subtraction set $S = \{1, 2, 4, 6\}$

   Let g_1, g_2, g_3 be the Sprague-Grundy functions of the 3 games respectively. Let G be the sum of the three games and g be the Sprague-Grundy function of G.
   (a) Write down the values of $g_1(7)$, $g_2(19)$ and $g_3(15)$
   (b) Find $g(7, 19, 15)$.
   (c) Find all winning moves of G from the position $(7, 19, 15)$.

4. (8 marks) Let

$$A = \begin{pmatrix} 3 & 1 & 2 & -2 & 0 \\ 2 & 3 & 1 & -3 & -1 \\ -1 & 2 & 0 & 4 & 1 \end{pmatrix}$$

(a) Write down the reduced matrix obtained by deleting all dominated rows and columns of A.

(b) Use the reduced matrix to solve the two-person zero sum game with game matrix A, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

5. (8 marks) Use simplex method to solve the game with the following game matrix, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

6. (8 marks) Let $a_1, a_2, \ldots, a_n > 0$ be positive real numbers and consider the zero sum game with $n \times n$ game matrix

$$A = \begin{pmatrix} a_1 & -a_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & a_2 & -a_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a_3 & -a_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & a_{n-2} & -a_{n-2} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & a_{n-1} & -a_{n-1} \\ -a_n & 0 & \cdots & \cdots & \cdots & 0 & a_n \end{pmatrix}$$

(a) Find a probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T = \mathbf{0}$ where $\mathbf{0} \in \mathcal{P}^n$ is the zero vector.

(b) Use the principle of indifference, or otherwise, to find a maximin strategy for the row player.

(c) Prove or disprove the following statement: If B is a 3×3 matrix and there exists a probability vector $\mathbf{y} \in \mathcal{P}^3$ such that $B\mathbf{y} = \mathbf{0}$, then the value of B is 0.

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