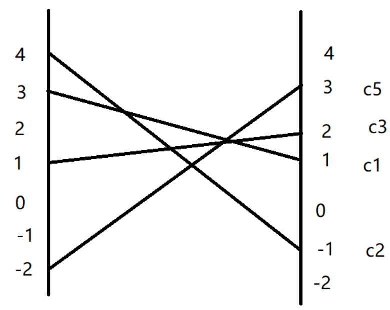
MATH4250 : Suggested Solution to Mid-Term

March 7, 2019

1. The set of P-position of the game is $\{(x, y) = (\lfloor n\varphi \rfloor, \lfloor n\varphi \rfloor + n), n \in \mathbb{N}, \varphi = \frac{1+\sqrt{5}}{2}\}$
   (a) As discussed above, only when $n = 6$, $\lfloor n\varphi \rfloor + n = 15$ could be satisfied.
   (b) In this cass, $n = 60$. Then $b = 97$
   (c) For $(15, 23)$, $\{(15, 9), (14, 23), (12, 20)\}$
   For $(100, 160)$, $\{(97, 157)\}$

2. (a) $g_1(11) = 3, g_1(12) = 4, g_1(13) = 4$
   (b) $g(16, 13, 7)$
   $= g_1(16) \oplus g_2(13) \oplus g_3(7)$
   $= 5 \oplus 6 \oplus 7$
   $= 4$
   (c) $6 \oplus 7 = 1$
   Then $(n, 13, 7)$ is a winning move if $g_1(n) = 1$ and $16 - n \geq 3$. Thus n could be $3, 4, 5$.
   $5 \oplus 7 = 2$
   Then $(16, n, 7)$ is a winning move if $g_2(n) = 2$ and $1 \geq 13 - n \leq 6$. Thus n could be 9.
   $5 \oplus 6 = 3$
   Then $(16, 13, n)$ is a winning move if $g_3(n) = 3$ and $n \leq 7$. Thus n could be 3.

3. (a) $\begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$



(b) Let $(x, 1-x)$ be maximum strategy for Aaron

$$\begin{pmatrix} x & 1-x \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 4-6x & 4x-1 \end{pmatrix}$$

Then $4 - 6x = 4x - 1$.
Then we have $x = \frac{1}{2}$, the maximum strategy is $(\frac{1}{2}, \frac{1}{2})$.
Let $(y, 1-y)$ be minimum strategy for Aaron

$$\begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} -5y+3 \\ 5y-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then we have $y = \frac{2}{5}$, the minimum strategy is $(\frac{2}{5}, \frac{3}{5})$

4. (a) $\begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{pmatrix}$

(b) $c_1 : v = x + 3(1-x)$
$c_2 : v = -x + 4(1-x)$
$c_3 : v = 2x + (1-x)$
$c_5 : v = 3x - 2(1-x)$
Intersection of C_5 and C_2, we have

$$v = -x + 4(1-x) \qquad = -5x + 4$$
$$v = 3x - 2(1-x) \qquad = 5x - 2$$

Then we get

$$x = \frac{3}{5}$$
$$v = 1$$

Then the max. strategy is $(0, \frac{3}{5}, \frac{2}{5})$. For the min. strategy:

$$\begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} y_2 \\ y_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then solve this equation, we have $y_2 = \frac{3}{5}$,$y_3 = \frac{2}{5}$
Then the max. strategy is $(0, \frac{1}{2}, 0, 0, \frac{1}{2})$. The value of game is $v = 1$

5.

$$\begin{array}{ccc} 3 & 1 & -2 \\ 2 & 3 & -1 \\ -1 & -2 & 2 \end{array} \rightarrow \begin{array}{ccc} 5 & 3 & 0 \\ 4 & 5 & 1 \\ 1 & 0 & 4 \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} 5 & 3 & 0 & 1 & 0 & 0 & 1 \\ 4 & 5 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 4 & 0 & 0 & 1 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} 1 & 3/5 & 0 & 1/5 & 0 & 0 & 1/5 \\ 1 & 5/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 1 & 0 & 4 & 0 & 0 & 1 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} 1 & 3/5 & 0 & 1/5 & 0 & 0 & 1/5 \\ 0 & 13/5 & 1 & -4/5 & 1 & 0 & 1/5 \\ 0 & -3/5 & 4 & -1/5 & 0 & 1 & 4/5 \\ \hline 0 & -2/5 & -1 & 1/5 & 0 & 0 & 1/5 \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} 1 & 2 & -3/13 & 5/13 & -3/13 & 0 & 2/13 \\ 0 & 1 & 5/13 & -4/13 & 5/13 & 0 & 1/13 \\ 0 & 0 & 55/13 & -5/13 & 3/13 & 1 & 11/13 \\ \hline 0 & 0 & 0 & 0 & 1/5 & 1/5 & 2/5 \end{array}$$

$d = \frac{2}{5}$
Max. Strategy : $q = \frac{1}{d}(y_1, y_2, y_3) = \frac{5}{2}(\frac{1}{5}, 0, \frac{1}{5}) = (\frac{1}{2}, 0, \frac{1}{2})$

Min. Strategy : $p = \frac{1}{d}(x_1, x_2, x_3) = \frac{5}{2}(0, \frac{1}{5}, \frac{1}{5}) = (0, \frac{1}{2}, \frac{1}{2})$
The value of the game : $v = \frac{1}{d} - k = \frac{1}{2}$

6. (a)

$$\begin{array}{ccccc} \left(\begin{array}{cc} 4k-3 & -(4k-2) \\ -(4k-1) & 4k \end{array}\right) & \begin{array}{c} 8k-5 \\ -(8k-1) \end{array} & \times & \begin{array}{c} 8k-1 \\ 8k-5 \end{array} & \begin{array}{c} \frac{8k-1}{16k-6} \\ \frac{8k-5}{16k-6} \end{array} \\ \begin{array}{ccc} 8k-4 & & -(8k-2) \\ & \times & \\ 8k-2 & & 8k-4 \\ \frac{8k-2}{16k-6} & & \frac{8k-4}{16k-6} \end{array} & & & & \end{array}$$

The max. strategy of A_k is $p = (\frac{8k-1}{16k-6}, \frac{8k-5}{16k-6})$
The min. strategy of A_k is $q = (\frac{8k-2}{16k-6}, \frac{8k-4}{16k-6})$
The value of this game is

$$v = \frac{(4k-3)(4k) - (4k-2)(4k-1)}{4k-3+4k+(4k-2)+(4k-1)} = \frac{1}{3-8k}$$

(b) Suppose $\hat{p} = (p_1, ..., p_n)$ is the maximum strtegy of D.
Then $\hat{p}A = (p_1/r_1, ..., p_n/r_n)$.
By principle of indifference, ${}_1/r_1 = ... = p_n/r_n = v$.
Then $p_i = r_i v$.
Also, $p_1 + ... + p_n = 1$. Then $v(r_1 + ... + r_n) = 1$.
Then $v = \dfrac{1}{r_1 + ... + r_n}$

(c)

$$
\begin{aligned}
v &= \frac{1}{r_1 + ... + r_n} \\
&= \frac{1}{\sum\limits_{k=1}^{25} \frac{1}{A_k}} \\
&= \frac{1}{\sum\limits_{k=1}^{25} 3 - 8k} \\
&= -\frac{1}{2525}
\end{aligned}
$$