math1510e & h

Week 10-Day 1

Topics covered

- Partial fraction decomposition
- Trigonometric integrals
- t −substitution
- Introduction to Fundamental Theorem of Calculus

Partial fraction decomposition – Long Division

There is "dictionary" between "rational numbers" and "rational functions".

For rational numbers, there are two types, i.e. proper rational numbers and improper rational numbers.

Examples are: $3/4$ (proper rational number), $4/3$ (improper rational number).

Now, by long division, one can convert any improper rational number to become a proper rational number (plus) an integer.

E.g. $\frac{4}{3} = 1 + \frac{1}{3}$

In the same way, one can always use long division to convert an improper rational function (i.e. a rational function $\frac{p(x)}{q(x)}$ satisfying $\deg p(x) \geq \deg q(x)$) into the sum of a "polynomial" and a "proper rational function".)

Example:

$$\frac{x^3 + 2x + 1}{x^2 + 1} = x + \frac{x + 1}{x^2 + 1}$$

Here the red-colored term is a polynomial. The yellow-colored term is a "proper" rational function.

To compute integral of the form $\int \frac{p(x)}{q(x)} dx$, where $p(x)$ and $q(x)$ are polynomials and $\deg p(x) < \deg q(x)$, we first perform

Cases for Proper Rational Functions

To integrate rational functions, the above discussion tells us that we need only to consider "proper rational function" (because integration of polynomial is easy).

Now $\frac{p(x)}{q(x)}$ is "proper" and we have the following cases:

1. $q(x)$ has only simple, linear factors.
2. $q(x)$ has only simple irreducible factors.
3. $q(x)$ has repeated linear factors.
4. $q(x)$ has repeated irreducible quadratic factors.
5. Combination of 3 and 4 above.

> Proof of the five cases in "Step 1" makes uses of complex numbers – one can show that (if one allows for complex number solutions) any degree n polynomial has exactly n roots. Also, whenever $a + b\sqrt{-1}$ is a root, then $a - b\sqrt{-1}$ is also a root. Such pair, when multiplied, forms the "irreducible quadratic factor" $(x - (a + b\sqrt{-1})(a - b\sqrt{-1})$. (Note: In this discussion, we assume that a, b are real numbers.)

Now we deal with these 4 cases one by one, via examples:

1. $\frac{x+1}{(x-1)(x-3)(x+7)} = \frac{A_1}{x-1} + \frac{A_2}{x-3} + \frac{A_3}{x+7}$

2. $\frac{x+2}{(x^2+x+1)(x^2+1)} = \frac{A_1+B_1x}{x^2+x+1} + \frac{A_2+B_2x}{x^2+1}$,

3. $\frac{x+1}{(x-1)^2(x-3)^3} = \frac{A_1}{(x-1)^2} + \frac{A_2}{x-1} + \frac{B_1}{(x-3)^3} + \frac{B_2}{(x-3)^2} + \frac{B_3}{x-3}$

4. $\frac{x+2}{(x^2+x+1)^2(x^2+1)^3} = \frac{A_1+B_1x}{(x^2+x+1)^2} + \frac{A_2+B_2x}{x^2+x+1} + \frac{C_1+D_1x}{(x^2+1)^3} + \frac{C_2+D_2x}{(x^2+1)^2} + \frac{C_3+D_3x}{x^2+1}$

5. $\frac{x+2}{(x^2+x+1)^2(x-1)^2} = \frac{A_1+B_1x}{(x^2+x+1)^2} + \frac{A_2+B_2x}{x^2+x+1} + \frac{C_1}{(x-1)^2} + \frac{C_2}{x-1}$

A computational Example

(A good internet website with an interactive calculator for this is: http://www.emathhelp.net/calculators/algebra-2/partial-fraction-decomposition-calculator/)

Find partial fraction decomposition of $\frac{x+7}{(x^2+1)(x^2+x+1)^2}$.

Since the denominator has two irreducible quadratic factors, i.e. x^2+1 and x^2+x+1. The first of them is a “simple” quadratic factor, the other is a “repeated” one.

So the form of the partial fraction decomposition is

$$\frac{x+7}{(x^2+1)(x^2+x+1)^2}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}+\frac{Ex+F}{x^2+1}$$

Forming common denominator on the right-hand side gives:

$$\frac{x+7}{(x^2+1)(x^2+x+1)^2}$$
$$=\frac{(x^2+1)(x^2+x+1)(Ax+B)+(x^2+1)(Cx+D)+(x^2+x+1)^2(Ex+F)(x^2+1)(x^2+x+1)^2}{(x^2+1)(x^2+x+1)^2}$$

Comparing the numerators on the left-hand side and on the right-hand side gives

$$x+7=(x^2+1)(x^2+x+1)(Ax+B)+(x^2+1)(Cx+D)+(x^2+x+1)^2(Ex+F)$$

Expand right-hand side:

$$x+7=x^5A+x^5E+x^4A+x^4B+2x^4E+x^4F+2x^3A+x^3B+x^3C+3x^3E+2x^3F+x^2A+2x^2B+x^2D+2x^2E+3x^2F+xA+xB+xC+xE+2xF+B+D+F$$

Expanding the right-hand side and then collect up like terms (i.e. terms of the form $x^0, x^1, x^2, \cdots$, we get:

$$x+7=x^5(A+E)+x^4(A+B+2E+F)+x^3(2A+B+C+3E+2F)+x^2(A+2B+D+2E+3F)+x(A+B+C+E+2F)+B+D+F$$

Coefficients of the $x^0, x^1, x^2, \cdot$terms on the left-hand side and on the right-hand side should be equal, so we get the following system of equations:

$$A+E=0,$$
$$A+B+2E+F=0,$$
$$2A+B+C+3E+2F=0,$$
$$A+2B+D+2E+3F=0,$$
$$A+B+C+E+2F=1,$$
$$B+D+F=7$$

Solving it, we get that $A=1, B=8, C=7, D=6, E=-1, F=-7$.

Therefore,

$$\frac{x+7}{(x^2+1)(x^2+x+1)^2}=\frac{x+8}{x^2+x+1}+\frac{7x+6}{(x^2+x+1)^2}+\frac{(-1)x-7}{x^2+1}.$$

Next, we have

Step 2

Integrate term by term the expressions obtained in the partial fraction decomposition.

Take our five examples again, we have

1. $\frac{x+1}{(x-1)(x-3)(x+7)} = \frac{A_1}{x-1} + \frac{A_2}{x-3} + \frac{A_3}{x+7}$

2. $\frac{x+2}{(x^2+x+1)(x^2+1)} = \frac{A_1+B_1x}{x^2+x+1} + \frac{A_2+B_2x}{x^2+1}$,

3. $\frac{x+1}{(x-1)^2(x-3)^3} = \frac{A_1}{(x-1)^2} + \frac{A_2}{x-1} + \frac{B_1}{(x-3)^3} + \frac{B_2}{(x-3)^2} + \frac{B_3}{x-3}$

4. $\frac{x+2}{(x^2+x+1)^2(x^2+1)^3} = \frac{A_1+B_1x}{(x^2+x+1)^2} + \frac{A_2+B_2x}{x^2+x+1} + \frac{C_1+D_1x}{(x^2+1)^3} + \frac{C_2+D_2x}{(x^2+1)^2} + \frac{C_3+D_3x}{x^2+1}$

5. $\frac{x+2}{(x^2+x+1)^2(x-1)^2} = \frac{A_1+B_1x}{(x^2+x+1)^2} + \frac{A_2+B_2x}{x^2+x+1} + \frac{C_1}{(x-1)^2} + \frac{C_2}{x-1}$

For 1, after integrating the right-hand side, we get $A_1 \ln|x-1| + A_2 \ln|x-3| + A_3 \ln|x+7| + C$. Similar result holds for 3.

For 2 and 4, use simple substitution and trigonometric substitution.

Trigonometric Integrals

The next topic we discussed briefly is “trigonometric integrals”. In many scientific disciplines, one needs to compute integrals of the form

- $\int f(x)\cos(nx)\ dx$,
- $\int f(x)\sin(mx)\ dx$,
- $\int \sin^n x \cos^m x\ dx$

One reason for this, especially for the first two types of integrals is due to something known as “Fourier series” which we briefly outline in the Appendix.:

Also important in Engineering are the integrals of the form

$$\int \cos^{\mathrm{n}} x \sin^{\mathrm{m}} x\ dx$$

We now discuss how to compute this kind of integrals.

There are two cases

1) n or m is odd number. In this case, we can single out one copy of $\sin x$ or $\cos x$ and get $d(-\cos x)$ or $d\sin x$.g. $\int \sin^2 x \cos^3 x\ dx = \int \sin^2 x \cos^2 x\ d\cos x$
2) If both n and m are even numbers, then we try to make use of the following double-angle formulas to “reduce” the “degree” of $\sin x$ or $\cos x$ by “one”.

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Two Examples:

1) $\int \sin^2 x \cos^3 x = \int \sin^2 x \cos^2 x \ d \sin x$

(Note that the term "$d \sin x$" comes from part of the term $\cos^3 x \, dx$)

Having done this, then we have

$\int \sin^2 x \cos^2 x \ d \sin x = \int \sin^2 x \ (1 - \sin^2 x) d \sin x = \int u^2 (1 - u^2) du$

This can be easily integrated.

2) $\int \sin^2 x \cos^4 x \ dx = \int \sin^2 x \ (\cos^2 x)^2 dx = \int (\sin x \cos x)^2 (\cos^2 x) dx$

$$= \int \sin^2(2x) \left(\frac{1 + \cos(2x)}{2} \right) dx = \cdots$$

(Note that we have managed to rewrite "integration" of "two EVEN powers" to "integration of one EVEN power, i.e. $\int \sin^2(2x) dx$" and "integration of one ODD and one EVEN power, i.e. $\int \sin^2(2x) \cos(2x) dx$ Proceeding in this way will finally lead to the answer).

$t-$ Substitution

This is a very clever substitution invented by Weierstrass. This substitution is for computing indefinite integral of the form:

$$\int \text{rational function of} \sin x \,\&\, \cos x \ dx$$

Examples:

1) $\int \frac{\sin x + 2 \cos x}{3 \sin x - 5 \cos x} \ dx$

2) $\int \frac{dx}{2 + \sin x}$

In such cases, the idea is to let $t = \tan\left(\frac{x}{2}\right)$.

Then we have

$$dx = \frac{2dt}{1 + t^2}$$

by direct computation of $\frac{dt}{dx}$ and using the formula $1 + \tan^2 x = \sec^2 x$.

And by considering a right-angled triangle of base length 1, height t, hypotenuse length $\sqrt{1 + t^2}$, angle $x/2$, we obtain the formulas

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}.$$

Using them, one can "rewrite" a rational function of "sine" and "cosine" as a rational function of t.

Example:

$$\int \frac{\sin x}{\sin x + 2\cos x} dx = \int \frac{\frac{2t}{1+t^2}}{\frac{2t}{1+t^2} + 2(\frac{1-t^2}{1+t^2})} \cdot \frac{2dt}{1+t^2}$$

Now one can use "partial fraction decomposition" to compute

$$\int \frac{\frac{2t}{1+t^2}}{\frac{2t}{1+t^2} + 2(\frac{1-t^2}{1+t^2})} \cdot \frac{2dt}{1+t^2}$$

Appendix (Fourier Series)

Given a periodic function, i.e. a function $f(x)$ satisfying $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, one has the following "representation" (The yellow-colored formula simply means "after time T, the value of the function $f(x+T)$ is the same as that at x, i.e. $f(x)$.)

Such function can be represented as a "sum" of "more elementary periodic functions", i.e. the sine and the cosine functions in the following way.

$$f(x) \text{ "="} a_0 + a_1 \cos x + a_2 \cos 2x + \cdots \ + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

The following webpage has more information if you are interested in this topic:
http://www.intmath.com/fourier-series/fourier-graph-applet.php

Remark:

We put "quote and quote" on the equality sign, because this equality is something special. Here, the main idea here is that a "periodic" function should be "representable" by some elementary "building blocks" which are also "periodic". Now we know that the sine, cosine functions are "periodic". Indeed, they form the basic building blocks of any periodic functions.

To compute the numbers $a_0, a_1, \cdots, a_n, \cdots$ we need to compute integrals of the form

$$\int f(x) \cos nx \, dx, \int f(x) \sin mx \, dx$$

They can be computed by, for example, using integration by parts.