

Methods of Integration

1.1 Basic formulas

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int e^x dx = e^x + C;$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int \cos x dx = \sin x + C;$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C;$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C;$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \tan x dx = \ln |\sec x| + C;$$

$$\int \cot x dx = \ln |\sin x| + C$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C; \quad \int \csc x dx = \ln |\csc x - \cot x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C;$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \cos^{-1} \frac{a}{x} + C$$

1.2 Trigonometric Integrals

Trigonometric identities:

1.
 - $\cos^2 x + \sin^2 x = 1$
 - $\sec^2 x = 1 + \tan^2 x$
 - $\csc^2 x = 1 + \cot^2 x$
2.
 - $\cos^2 x = \frac{1 + \cos 2x}{2}$
 - $\sin^2 x = \frac{1 - \cos 2x}{2}$
 - $\cos x \sin x = \frac{\sin 2x}{2}$
3.
 - $\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y))$
 - $\cos x \sin y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$
 - $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$

Integral of the form $\int \cos^m x \sin^n x dx$ where m, n are non-negative integers,

Case 1. If m is odd, use $\cos x dx = d \sin x$. (Substitute $u = \sin x$.)

Case 2. If n is odd, use $\sin x dx = -d \cos x$. (Substitute $u = \cos x$.)

Case 3. If both m, n are even, then use double angle formulas to reduce the power.

$$\bullet \cos^2 x = \frac{1 + \cos 2x}{2} \quad \bullet \sin^2 x = \frac{1 - \cos 2x}{2} \quad \bullet \cos x \sin x = \frac{\sin 2x}{2}$$

Integral of the form $\int \sec^m x \tan^n x dx$ where m, n are non-negative integers,

Case 1. If m is even, use $\sec^2 x dx = d \tan x$. (Substitute $u = \tan x$.)

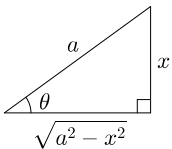
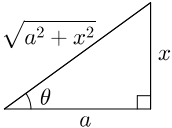
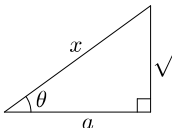
Case 2. If n is odd, use $\sec x \tan x dx = d \sec x$. (Substitute $u = \sec x$.)

Case 3. If both m is odd and n is even, use $\tan^2 x = \sec^2 x - 1$ to write everything in terms of $\sec x$.

1.3 Integration By Parts

$$\int u dv = uv - \int v du$$

1.4 Trigonometric Substitution

Expression	Substitution	dx	Trigonometric ratios
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$dx = a \cos \theta d\theta$	 $\cos \theta = \frac{\sqrt{a^2 - x^2}}{a}$ $\sin \theta = \frac{x}{a}$ $\tan \theta = \frac{x}{\sqrt{a^2 - x^2}}$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$dx = a \sec^2 \theta d\theta$	 $\cos \theta = \frac{a}{\sqrt{a^2 + x^2}}$ $\sin \theta = \frac{x}{\sqrt{a^2 + x^2}}$ $\tan \theta = \frac{x}{a}$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$dx = a \sec \theta \tan \theta d\theta$	 $\cos \theta = \frac{a}{x}$ $\sin \theta = \frac{\sqrt{x^2 - a^2}}{x}$ $\tan \theta = \frac{\sqrt{x^2 - a^2}}{a}$

1.5 Integration of Rational Functions

Any rational function $R(x) = \frac{f(x)}{g(x)}$ can be expressed in partial fraction of the form

$$R(x) = q(x) + \sum \frac{A}{(x - \alpha)^k} + \sum \frac{B(x + a)}{((x + a)^2 + b^2)^k} + \sum \frac{C}{((x + a)^2 + b^2)^k}$$

Partial fractions can be integrated using the formulas below.

$$\begin{aligned} \bullet \int \frac{dx}{(x - \alpha)^k} &= \begin{cases} \ln|x - \alpha| + C, & \text{if } k = 1 \\ -\frac{1}{(k - 1)(x - \alpha)^{k-1}} + C, & \text{if } k > 1 \end{cases} \\ \bullet \int \frac{x dx}{(x^2 + a^2)^k} &= \begin{cases} \frac{1}{2} \ln(x^2 + a^2) + C, & \text{if } k = 1 \\ -\frac{1}{2(k - 1)(x^2 + a^2)^{k-1}} + C, & \text{if } k > 1 \end{cases} \\ \bullet \int \frac{dx}{(x^2 + a^2)^k} &= \begin{cases} \frac{1}{a} \tan^{-1} \frac{x}{a} + C, & \text{if } k = 1 \\ \frac{x}{2a^2(k - 1)(x^2 + a^2)^{k-1}} + \frac{2k - 3}{2a^2(k - 1)} \int \frac{dx}{(x^2 + a^2)^{k-1}}, & \text{if } k > 1 \end{cases} \end{aligned}$$

1.6 t -substitution

To evaluate

$$\int R(\cos x, \sin x, \tan x) dx$$

where R is a rational function, we may use t -substitution

$$t = \tan \frac{x}{2}.$$

Then

$$\tan x = \frac{2t}{1 - t^2}; \quad \cos x = \frac{1 - t^2}{1 + t^2}; \quad \sin x = \frac{2t}{1 + t^2};$$

$$dx = d(2 \tan^{-1} t) = \frac{2dt}{1 + t^2}.$$

We have

$$\int R(\cos x, \sin x, \tan x) dx = \int R\left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}, \frac{2t}{1 - t^2}\right) \frac{2dt}{1 + t^2}$$

which is an integral of rational function.